

# COMPLETENESS AND SPECTRAL SYNTHESIS OF NONSELFADJOINT ONE-DIMENSIONAL PERTURBATIONS OF SELFADJOINT OPERATORS

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**ABSTRACT.** A (linear unbounded) operator  $\mathcal{L}$  is called a finite-dimensional singular perturbation of an operator  $\mathcal{A}$  if their graphs differ in a finite-dimensional space. We study spectral properties of a one-dimensional singular perturbation of an unbounded selfadjoint operator. Our approach is based on a functional model for this operator. We give criteria for the completeness of the perturbed operator and study the relation between completeness of the perturbed operator and its adjoint. It is shown that even for the case of rank one perturbations the conditions of the Keldyš and Macaev completeness theorems are sharp. In particular, there exist rank one perturbations with trivial kernels which are complete while their adjoints are not. Most of our results have their counterparts for rank one perturbations of compact selfadjoint operators.

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## 0. INTRODUCTION AND MAIN RESULTS

**0.1. Preliminary remarks.** In this paper, all Hilbert spaces will be supposed to be separable. A Banach or Hilbert space (linear) operator will be called *complete* if it has a complete set of eigenvectors and root vectors. Despite a large effort devoted to the study of criteria of completeness of operators, our understanding still is far from perfect, even for the case of ordinary differential operators. Most general abstract results are due to Keldyš [41, 42] and Macaev [59] (see, also, [28, Chapter V]). We recall that an operator  $S$  on a Hilbert space belongs to the Macaev ideal  $\mathfrak{S}_\omega$  if it is compact and its singular numbers  $s_k$  satisfy the relation  $\sum_{k \geq 1} \frac{s_k}{k} < \infty$ .

**Theorem** (Keldyš, 1951). *Suppose  $\mathcal{A}$  is a selfadjoint Hilbert space operator that belongs to a Schatten ideal  $\mathfrak{S}_p$ ,  $0 < p < \infty$  and satisfies  $\ker \mathcal{A} = 0$ . Let  $\mathcal{L} = \mathcal{A}(I + S)$ , where  $S$  is compact and  $\ker(I + S) = 0$ . Then the operators  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.*

**Theorem** (Macaev, 1961). *If  $\mathcal{L} = \mathcal{A}(I + S)$ , where  $\mathcal{A}$ ,  $S$  are compact operators on a Hilbert space,  $\mathcal{A}$  is selfadjoint,  $S \in \mathfrak{S}_\omega$  and  $\ker \mathcal{A} = \ker(I + S) = 0$ , then  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.*

A proof of this theorem and its generalizations to operator pencils can be found in [61]. As Macaev proved in [60, Theorem 3], in the above result  $\mathfrak{S}_\omega$  can be replaced by wider class, depending on  $\mathcal{A}$ . Perturbations of a selfadjoint compact operator  $\mathcal{A}$  that have the form  $\mathcal{A}(I + S)$  or  $(I + S)\mathcal{A}$ , where  $S$  is compact, are called *weak perturbations*. In [63], Macaev and Mogul'skii give an explicit condition on the spectrum of  $\mathcal{A}$ , equivalent to the property that all weak perturbations of  $\mathcal{A}$  are complete; see also [62].

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In this article, we consider a very concrete class of rank  $n$  perturbations of compact self-adjoint operators, which are not necessarily weak. Our main results are concerned with rank 1 (nondissipative) perturbations of compact selfadjoint operators. Applying a functional model for these operators close to a model due to Kapustin [38], we give some new criteria for completeness of operators of this class as well as some new (counter)examples. In particular, we will show the sharpness of the conditions of Macaev's theorem even for this class.

We also consider what we call *rank one singular perturbations* of unbounded selfadjoint operators with discrete spectrum and obtain parallel completeness results for them.

**0.2. The definition of singular perturbations of unbounded operators.** We begin with the following definition, motivated by [6], where a more general case of closed linear relations was treated. We refer to [20], [5], [69], [15] and to books [30], [51], [48] for alternative treatments of singular perturbations in more general settings.

**Definition.** Let  $\mathcal{A}, \mathcal{L}$  be unbounded closed linear operators on a Banach space  $H$ . We say that  $\mathcal{L}$  is a finite rank singular perturbation of  $\mathcal{A}$  if their graphs  $G(\mathcal{A}), G(\mathcal{L})$  in  $H \oplus H$  differ in a finite dimensional space. If, moreover,

$$(0.1) \quad \dim(G(\mathcal{A})/(G(\mathcal{A}) \cap G(\mathcal{L}))) = \dim(G(\mathcal{L})/(G(\mathcal{A}) \cap G(\mathcal{L}))) = n < \infty,$$

then we will say that  $\mathcal{L}$  is a balanced rank  $n$  singular perturbation of  $\mathcal{A}$ .

If  $\mathcal{A}$  and  $\mathcal{L}$  are differential operators, defined by the same regular differential expression of order  $n$  and different boundary relations ( $n$  independent relations in both cases), then  $\mathcal{L}$  is a balanced singular perturbation of  $\mathcal{A}$  of rank less or equal to  $n$ .

The main questions we study in this paper are the following:

*Given a singular rank one perturbation  $\mathcal{L}$  of a cyclic selfadjoint operator  $\mathcal{A}$ , when are operators  $\mathcal{L}$  and  $\mathcal{L}^*$  complete? Under which assumptions completeness of  $\mathcal{L}$  implies completeness of  $\mathcal{L}^*$ ?*

In Section 1, we will relate this question with the completeness of bounded rank  $n$  perturbations of a compact selfadjoint operator and therefore with theorems by Keldyš and Macaev. In fact, Proposition 1.5 below implies that, roughly speaking, rank  $n$  singular perturbations of a selfadjoint operator  $\mathcal{A}$  with discrete spectrum are inverses of rank  $n$  (bounded) perturbations of  $\mathcal{A}^{-1}$ .

Suppose  $\mathcal{A}$  is a closed, densely defined linear operator on a Hilbert space  $H$ . We denote by  $\sigma(\mathcal{A})$  the spectrum of  $\mathcal{A}$  and by  $\rho(\mathcal{A}) = \mathbb{C} \setminus \sigma(\mathcal{A})$  its resolvent set. In what follows we will always assume that  $0 \in \rho(\mathcal{A})$ . An obvious modification of our construction below works if  $\rho(\mathcal{A}) \neq \emptyset$ .

We define the Hilbert space  $\mathcal{A}H$  as the set of formal expressions  $\mathcal{A}x$ , where  $x$  ranges over the whole space  $H$ . Put  $\|\mathcal{A}x\|_{\mathcal{A}H} = \|x\|_H$  for all  $x \in H$ . The formula  $x = \mathcal{A}(\mathcal{A}^{-1}x)$  allows one to interpret  $H$  as a linear submanifold of  $\mathcal{A}H$ . We consider the scale of spaces

$$\mathcal{D}(\mathcal{A}) \subseteq H \subseteq \mathcal{A}H.$$

Under our assumptions,  $\mathcal{A}^*$  is well defined, and there are natural identifications  $\mathcal{D}(\mathcal{A}^*) = (\mathcal{A}H)^*$ ,  $\mathcal{D}(\mathcal{A}) = (\mathcal{A}^*H)^*$ .

The rank  $n$  singular balanced perturbations of  $\mathcal{A}$  can be described in terms of what we will call  $n$ -data for  $\mathcal{A}$ . Let  $n \in \mathbb{N}$ . By  $n$ -data we mean a triple  $(a, b, \varkappa)$ , where

$$(0.2) \quad a : \mathbb{C}^n \rightarrow \mathcal{A}H, \quad b : \mathbb{C}^n \rightarrow \mathcal{A}^*H, \quad \varkappa : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

are linear and bounded,  $\text{rank } a = \text{rank } b = n$ , and for any  $c \in \mathbb{C}^n$ ,

$$\mathcal{A}^{-1}ac \in \mathcal{D}(\mathcal{A}), \quad \varkappa c = b^*(\mathcal{A}^{-1}ac) \implies c = 0. \quad (\mathbf{A}_n)$$

Notice that  $b^*$  is an operator  $b^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{C}^n$ .

For any  $n$ -data  $(a, b, \varkappa)$ , we define a linear operator  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  in the following way:

$$(0.3) \quad \begin{aligned} \mathcal{D}(\mathcal{L}) &\stackrel{\text{def}}{=} \{y = y_0 + \mathcal{A}^{-1}ac : \\ &c \in \mathbb{C}^n, y_0 \in \mathcal{D}(\mathcal{A}), \varkappa c + b^*y_0 = 0\}; \\ \mathcal{L}(\mathcal{A}, a, b, \varkappa)y &\stackrel{\text{def}}{=} \mathcal{A}y_0, \quad y \in \mathcal{D}(\mathcal{L}). \end{aligned}$$

Condition  $(A_n)$  is equivalent to the uniqueness of the decomposition  $y = y_0 + \mathcal{A}^{-1}ac$  for  $y \in \mathcal{D}(\mathcal{L}(\mathcal{A}, a, b, \varkappa))$  and hence to the correctness of the definition of  $\mathcal{L}$ .

The introduction of this kind of perturbation is justified by the following observation. It is easy to see that for each  $n$ -data  $(a, b, \varkappa)$ , the above-defined operator  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is a balanced rank  $n$  singular perturbation of  $\mathcal{A}$ . Moreover, any balanced rank  $n$  singular perturbation of  $\mathcal{A}$  has the form  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  for some  $n$ -data, which, in a sense, is determined uniquely (see Proposition 1.2 below). It would be interesting to know whether this parametrization of balanced rank  $n$  singular perturbations extends to a wider class of compact singular perturbations [6].

There are another ways of introducing singular perturbations, that permit one to deal with infinite rank perturbations, see [51], [72] and references therein.

**0.3. A model for singular rank one perturbations.** One of the main objects of our study are singular rank one perturbations of an unbounded selfadjoint operator  $\mathcal{A}$ , given by

$$(0.4) \quad H \stackrel{\text{def}}{=} L^2(\mu), \quad (\mathcal{A}f)(x) = xf(x), \quad x \in L^2(\mu).$$

By the spectral theorem, any selfadjoint operator of spectral multiplicity 1 can be represented in this form. We assume  $\mu$  to be a *singular measure* and  $\text{supp } \mu \neq \mathbb{R}$ . Without loss of generality we may assume then that  $0 \notin \text{supp } \mu$ .

Notice that for  $\mathcal{A}$  given in this form,  $\mathcal{A}H = xL^2(\mu)$ . Hence the 1-data for  $\mathcal{A}$  is just a triple  $(a, b, \varkappa)$ , where

$$(0.5) \quad \frac{a}{x}, \frac{b}{x} \in L^2(\mu); \quad \varkappa \in \mathbb{C}$$

(however, possibly,  $a, b \notin L^2(\mu)$ ) and the condition

$$\varkappa \neq \int_{\mathbb{R}} x^{-1} a(x) \overline{b(x)} d\mu(x) \tag{A}$$

in the case when  $a(x) \in L^2(\mu)$

is fulfilled. The corresponding singular perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  of  $\mathcal{A}$  is defined as follows:

$$(0.6) \quad \begin{aligned} \mathcal{D}(\mathcal{L}) &\stackrel{\text{def}}{=} \{y = y_0 + c \cdot \mathcal{A}^{-1}a : \\ &c \in \mathbb{C}, y_0 \in \mathcal{D}(\mathcal{A}), \varkappa c + \langle y_0, b \rangle = 0\}; \\ \mathcal{L}y &\stackrel{\text{def}}{=} \mathcal{A}y_0, \quad y \in \mathcal{D}(\mathcal{L}). \end{aligned}$$

Throughout the paper, in the case of a rank one perturbation, we make the following

**Assumption.** The function  $a \in (1 + |x|)^{-1}L^2(\mu)$  is nonzero and  $b \in (1 + |x|)^{-1}L^2(\mu)$  is a *cyclic* vector for the resolvent of  $\mathcal{A}$ , that is,  $b \neq 0$   $\mu$ -a.e.

We will also make the same assumption in the case of a bounded perturbation  $\mathcal{A} + ab^*$  of a bounded operator  $\mathcal{A}$ . Here and in what follows, for  $a, b \in H$ , we denote by  $ab^*$  the rank one operator  $ab^*x = (x, b)a$ .

In Lemma 2.4 below, we give a formula for  $\sigma(\mathcal{L})$  in terms of the zero set of an analytic function  $\varphi$ , defined explicitly in terms of the data  $(a, b, \varkappa)$ .

Most of our results will concern the case when  $\mu$  is a discrete measure:  $\mu = \sum_{n \in \mathcal{N}} \mu_n \delta_{t_n}$ , where  $\lim_{|n| \rightarrow \infty} |t_n| = \infty$ . The spectrum of  $\mathcal{A}$  is

$$\sigma(\mathcal{A}) = \{t_n : n \in \mathcal{N}\}$$

and has no accumulation points on  $\mathbb{R}$ . In this situation, we will assume that  $\{t_n\}$  is strictly increasing, and the index set  $\mathcal{N}$  can be  $\mathbb{N}$ ,  $-\mathbb{N}$  or  $\mathbb{Z}$  (for doubly infinite sequences). Then we will say that  $\mathcal{A}$  has a discrete spectrum and will use the notation

$$a_n = a(t_n), \quad b_n = b(t_n).$$

**Definition.** We call  $\mathcal{L}$  a real type perturbation of  $\mathcal{A}$  if the function  $\bar{a}b$  and the number  $\varkappa$  are real. We call  $\mathcal{L}$  a strong real type perturbation if, in addition to the above two requirements, the spectrum  $\sigma(\mathcal{L})$  is contained in  $\mathbb{R}$ .

The spectrum of any real type perturbation of  $\mathcal{A}$  is symmetric with respect to the real line. Recently there have been much interest in the study of nonselfadjoint operators with real spectrum in connection with non-Hermitian Hamiltonians that have a space-time reflection symmetry, see [4], [12] and references therein.

We recall that for any closed operator  $\mathcal{L}$  on  $H$ , the adjoint operator  $\mathcal{L}^*$  is well-defined if and only if  $\mathcal{D}(\mathcal{L})$  is dense in  $H$ . Let  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be any singular rank one perturbation of  $\mathcal{A}$ . It will follow from Proposition 1.3 that  $\mathcal{L}^*$  exists whenever

$$\varkappa \neq \int_{\mathbb{R}} x^{-1} a(x) \overline{b(x)} d\mu(x) \quad (\text{A}^*)$$

in the case when  $b \in L^2(\mu)$ .

Moreover, if (A\*) holds, then  $\mathcal{L}^* = \mathcal{L}(\mathcal{A}, b, a, \varkappa^*)$ .

**0.4. Main results.** In Section 1, we give some background on rank  $n$  singular perturbations of selfadjoint operators with discrete spectrum and relate them with rank  $n$  (bounded) perturbations of selfadjoint compact operators. In Propositions 1.1 and 1.7, we give some completeness results for these two classes of operators by applying Macaev's theorem and a simple additional argument. As a corollary, we obtain the completeness for what we call *generalized weak perturbations*, i.e., the perturbations satisfying

$$(0.7) \quad \sum_{n \in \mathcal{N}} \frac{|a_n b_n| \mu_n}{|t_n|} < \infty, \quad \sum_{n \in \mathcal{N}} \frac{a_n \bar{b}_n \mu_n}{t_n} \neq \varkappa$$

(see Theorem 3.3 and Proposition 5.1).

The following statements are the main results of this work. In these statements,  $\mathcal{A}$  is assumed to be a cyclic selfadjoint operator with discrete spectrum without accumulation points on  $\mathbb{R}$ .

The first result shows that the property of being a generalized weak perturbation may be replaced by a positivity condition.

**Theorem 0.1.** Suppose  $\mathcal{A}$  has a discrete spectrum, and let  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be its singular real type rank one perturbation. Suppose that  $a_n \bar{b}_n \geq 0$  for all but possibly a finite number of values of  $n$  and  $\sum_{n \in \mathcal{N}} |t_n|^{-1} |a_n b_n| \mu_n = \infty$ . Then  $\mathcal{L}^*$  is correctly defined, and  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.

The next theorem shows that under certain additional assumptions the completeness of a singular rank one perturbation  $\mathcal{L}$  implies the completeness of its adjoint.

**Theorem 0.2.** *Assume that  $\mathcal{A}$  has a discrete spectrum and that the data  $(a, b, \varkappa)$  satisfy  $a \notin L^2(\mu)$  (and, thus, (A) holds). Assume also condition (A\*). Let the perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be complete. Then its adjoint  $\mathcal{L}^*$  is also complete if any of the following conditions is fulfilled:*

- (i)  $|a_n|^2 \mu_n \geq C|t_n|^{-N} > 0$  for some  $N > 0$ ;
- (ii)  $|b_n a_n^{-1}| \leq C|t_n|^{-N}$  for some  $N > 0$ .

In Theorem 3.3 conditions sufficient for the joint completeness of  $\mathcal{L}$  and  $\mathcal{L}^*$  are given in terms of the generating function  $\varphi$ , the parameter of the functional model.

In general, the situation is much subtler. Applying the results and methods from [8, 9] we are able to give several examples when the perturbed operator (or its adjoint) fails to be complete as soon as the Macaev or Keldyš type conditions are relaxed:

**Theorem 0.3.** *For any cyclic selfadjoint operator  $\mathcal{A}$  with discrete spectrum  $\{t_n\}_{n \in \mathbb{N}}$ , there exists a strong real type singular rank one perturbation  $\mathcal{L}$  of  $\mathcal{A}$ , which is not complete, while its adjoint  $\mathcal{L}^*$  is correctly defined, has trivial kernel and is complete. Moreover, the orthogonal complement to the space spanned by the eigenvectors of  $\mathcal{L}$  may be infinite-dimensional.*

Note that if a bounded operator  $T$  is complete, a trivial obstacle for completeness of  $T^*$  is that  $T$  may have a nontrivial kernel, while  $\ker T^* = 0$ . The first (highly nontrivial) examples of the situation where  $T$  is complete and  $\ker T = 0$ , while  $T^*$  is not complete, were constructed by Hamburger [33]. In [19] Deckard, Foias and Pearcy gave a simpler construction. However, in these examples one cannot conclude that the corresponding operator is a small (in some sense) perturbation of a selfadjoint operator. An analog of Theorem 0.3 for the compact case shows that one can find such examples among rank one perturbations of compact selfadjoint operators with arbitrary spectra.

**Corollary 0.4.** *For any compact selfadjoint operator<sup>1</sup>  $\mathcal{A}_0$  with simple point spectrum and trivial kernel there exists a bounded rank one perturbation  $\mathcal{L}_0$  of  $\mathcal{A}_0$  with real spectrum such that  $\mathcal{L}_0$  is complete and  $\ker \mathcal{L}_0 = 0$ , while  $\mathcal{L}_0^*$  is not complete.*

Another example is related to the spectral synthesis problem. Recall that a bounded linear operator  $T$  in a Hilbert space is said to *admit spectral synthesis* if any  $T$ -invariant subspace  $M$  coincides with the closed linear span of the eigenvectors and root vectors which belong to  $M$  (equivalently, the restriction of  $T$  to  $M$  is complete). Notice that the spectral synthesis holds for all normal compact operators. The first example of a compact operator which does not admit spectral synthesis also goes back to Hamburger [33]. Further examples and generalizations were obtained by Nikolski [66] and Markus [56].

It turns out that this phenomenon is possible even for rank one perturbations of compact selfadjoint operators.

**Theorem 0.5.** *Let  $\mathcal{A}_0$  be a compact selfadjoint operator with simple point spectrum  $\{s_n\}_{n \in \mathbb{N}}$ ,  $s_n \neq 0$  (ordered so that  $s_n$  are positive and decrease for  $n \geq 0$ , and  $s_n$  are negative and increase for  $n < 0$ ). Assume that for some  $C, N > 0$ ,*

$$|s_n|^N \leq C|s_{n+1} - s_n| = o(|s_n|), \quad |n| \rightarrow \infty.$$

*Then there exists a rank one perturbation  $\mathcal{L}_0$  of  $\mathcal{A}_0$  of strong real type such that both  $\mathcal{L}_0$  and  $\mathcal{L}_0^*$  have complete sets of eigenvectors, but  $\mathcal{L}$  does not admit spectral synthesis.*

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<sup>1</sup>We use the notation  $\mathcal{A}_0$ ,  $\mathcal{L}_0$  to distinguish the compact operators from the case of singular rank one perturbations of unbounded self-adjoint operators, which are denoted by  $\mathcal{A}$  and  $\mathcal{L}$ .



We also are able to prove that, under some restrictions on the spectrum and the perturbations, the spectral synthesis holds up to a finite-dimensional orthogonal complement (see Theorem 5.2).

Finally, using our techniques it is easy to show that even for rank one perturbations, when we relax slightly the generalized weakness property (0.7), the resulting perturbation may become the inverse to a *Volterra operator* with trivial kernel (we recall that an operator is called Volterra if it is compact and its spectrum equals  $\{0\}$ ). We state the corresponding result for perturbations of compact operators.

**Theorem 0.6.** *There exists a sequence  $s_n \rightarrow 0$  and a measure  $\mu = \sum_n \mu_n \delta_{s_n}$  with the following property: for any  $\alpha_1, \alpha_2 \geq 0$  with  $\alpha_1 + \alpha_2 < 1$  there exist  $a \in |x|^{\alpha_1} L^2(\mu)$  and  $b \in |x|^{\alpha_2} L^2(\mu)$  such that the perturbed operator  $\mathcal{L}_0 = \mathcal{A}_0 + ab^*$  (where  $\mathcal{A}_0$  is the operator of multiplication by  $x$  in  $L^2(\mu)$ ) is a Volterra operator satisfying  $\ker \mathcal{L}_0 = \ker \mathcal{L}_0^* = 0$ .*

It is shown in [62, 63] that if  $\mathcal{A}$  is a positive operator, whose spectrum is sufficiently dense, then there is a weak perturbation of  $\mathcal{A}$ , which is a Volterra operator.

In a forthcoming paper, we will study in more detail the following question: *For which measures  $\mu$  does there exist a singular rank one perturbation  $\mathcal{L}$  of  $\mathcal{A}$  of the above type, whose spectrum is empty?* This question is closely related to looking for a bounded rank one perturbation of  $\mathcal{A}^{-1}$ , which is a Volterra operator, see Proposition 1.5 below.

**0.5. Some remarks on our methods.** Let  $\mathbb{C}^\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$  denote the upper and the lower half-planes and set  $H^2 = H^2(\mathbb{C}^+)$  to be the Hardy space in  $\mathbb{C}^+$ . For the basic properties of the space  $H^2$  we refer to [34, 36].

Define

$$(0.8) \quad \beta(z) = \varkappa + zb^*(\mathcal{A} - z)^{-1} \mathcal{A}^{-1} a = \varkappa + \int \left( \frac{1}{x - z} - \frac{1}{x} \right) a(x) \overline{b(x)} d\mu(x),$$

$$(0.9) \quad \rho(z) = \delta + zb^*(\mathcal{A} - z)^{-1} \mathcal{A}^{-1} b = \delta + \int \left( \frac{1}{x - z} - \frac{1}{x} \right) |b(x)|^2 d\mu(x),$$

where  $\delta$  is an arbitrary real constant. Since  $\mu(\{0\}) = 0$ ,  $zb^*(\mathcal{A} - z)^{-1} \mathcal{A}^{-1} a \neq \text{const}$ , and therefore  $\beta \not\equiv 0$  in  $\mathbb{C} \setminus \mathbb{R}$ .

We set

$$(0.10) \quad \Theta(z) = \frac{i - \rho(z)}{i + \rho(z)},$$

$$(0.11) \quad \varphi(z) = \frac{\beta(z)}{2} (1 + \Theta(z)).$$

It is easy to see that  $\Theta$  and  $\varphi$  are analytic in  $\mathbb{C}^+$ . Since  $\mu$  is a singular measure on  $\mathbb{R}$ , it follows that  $\operatorname{Im} \rho(z) \geq 0$  for  $z \in \mathbb{C}^+$  and  $\operatorname{Im} \rho(z) = 0$  a.e. on  $\mathbb{R}$ . Therefore  $\Theta$  is an inner function in the upper half-plane  $\mathbb{C}^+$  (that is, a bounded analytic function with  $|\Theta| = 1$  a.e. on  $\mathbb{R}$  in the sense of nontangential boundary values). Therefore  $\Theta$  generates a *shift-covariant* or *model subspace*  $K_\Theta \stackrel{\text{def}}{=} H^2 \ominus \Theta H^2$  of the Hardy space  $H^2$ . These subspaces, as well as their vector-valued generalizations, play an outstanding role both in function theory and in operator theory. For their numerous applications we refer to [65].

The following statement will be our main tool for studying the rank one perturbations:

**Theorem 0.7** (a functional model). *Let  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be a singular rank one perturbation of  $\mathcal{A}$ , and define  $\Theta$  and  $\varphi$  by the above formulas. Then  $\Theta$  is analytic in a neighborhood*

of 0,  $1 + \Theta \notin H^2$ ,  $\Theta(0) \neq -1$ ,

$$(0.12) \quad \varphi \notin H^2, \quad \frac{\varphi(z) - \varphi(i)}{z - i} \in K_\Theta,$$

and  $\mathcal{L}$  is unitary equivalent to the operator  $T = T_{\Theta, \varphi}$  which acts on the model space  $K_\Theta \stackrel{\text{def}}{=} H^2 \ominus \Theta H^2$  by the formulas

$$\mathcal{D}(T) \stackrel{\text{def}}{=} \{f = f(z) \in K_\Theta : \text{there exists } c = c(f) \in \mathbb{C} : zf - c\varphi \in K_\Theta\},$$

$$Tf \stackrel{\text{def}}{=} zf - c\varphi, \quad f \in \mathcal{D}(T).$$

If, moreover,  $\mathcal{L}$  is a real type singular rank one perturbation, then

$$(0.13) \quad \Theta = \frac{\varphi}{\bar{\varphi}} \quad \text{a.e. on } \mathbb{R}.$$

Conversely, any inner function  $\Theta$  which is analytic in a neighborhood of 0 and satisfies  $1 + \Theta \notin H^2$ ,  $\Theta(0) \neq -1$ , and any function  $\varphi$  satisfying (0.12) correspond to some singular rank one perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  of the operator  $\mathcal{A}$  of multiplication by the independent variable in  $L^2(\mu)$ , where  $\mu$  is some singular measure on  $\mathbb{R}$  and  $x^{-1}a(x), x^{-1}b(x) \in L^2(\mu)$ . If, moreover,  $\Theta$  and  $\varphi$  satisfy (0.13), then the perturbation  $\mathcal{L}$  is of real type.

This model is close to Kapustin's model for rank one perturbations of singular unitary operators [38]. We refer to [39] for a more general construction.

Gubreev and Tarasenko in [32] constructed a model for operators that have a discrete spectrum (compact resolvent), are neither dissipative nor anti-dissipative, whose imaginary part is two-dimensional, under an additional restriction that their spectrum does not intersect the real axis (last restriction does not seem to be essential). If we suppose that  $a, b \in L^2(\mu)$ , then  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is a rank one bounded perturbation of  $\mathcal{A}$  (see §2 below), and it follows that  $\mathcal{L}$  has two-dimensional imaginary part and is neither dissipative nor anti-dissipative (unless  $\mathcal{L} = \mathcal{L}^*$ ). It is easy to see that, conversely, any operator  $\mathcal{A} + i(f_1 f_1^* - f_2 f_2^*)$  on  $H$ , where  $\mathcal{A} = \mathcal{A}^*$ ,  $f_j \in H$  ( $j = 1, 2$ ) can be represented as  $\mathcal{A}_1 + ab^*$ , where  $\mathcal{A}_1 = \mathcal{A}_1^*$  and  $a, b$  are linear combinations of  $f_1, f_2$ . Hence the class of operators we consider is very close to the class of operators in the paper [32], and in fact, their model is essentially the same as ours. Paper [32] also contains certain results on completeness and on Riesz bases of eigenvectors of the perturbed operator (later on, we will comment on the connections between these results and ours).

In [44], Kiselev and Naboko study a general operator with two-dimensional imaginary part by making use of the Naboko model. A related model for operators of this class that have real spectrum was constructed by Zolotarev in [83] in terms of certain generalization of the de Branges spaces. The main point in the works by Kiselev and Naboko [44], [45] and others is the study of so-called almost Hermitian non-dissipative operators; this is a stronger requirement than the assumption that the spectrum is real. By using [45, Theorem 3.1], it is easy to check that in the model given by the above theorem,  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is almost Hermitian if and only if  $\varphi$  is outer. The perturbations in Theorems 0.3 and 0.5 can be chosen to be almost Hermitian, as is seen from the proofs.

For a more general model based on operator-valued characteristic functions see [72].

In view of Theorem 0.7, the completeness of  $\mathcal{L}$  (or  $\mathcal{L}^*$ ) translates into the completeness of  $T$  or  $T^*$ , respectively.

In what follows, we use the term “meromorphic” in the sense “meromorphic in  $\mathbb{C}$ ”. In the case when  $\mathcal{A}$  has a discrete spectrum,  $\Theta$  and  $\varphi$  are meromorphic. Then any function in  $K_\Theta$  is also meromorphic in  $\mathbb{C}$ . As it will be explained in Section 2.4 below, this situation reduces to the study of de Branges spaces of entire functions.

In the case when  $\Theta$  is meromorphic the eigenfunctions of  $T$  are of the form  $\frac{\varphi}{z-\lambda}$ ,  $\lambda \in Z_\varphi$ , where  $Z_\varphi$  is the zero set for  $\varphi$  (see Lemma 2.4 below for a more accurate statement), while the eigenfunctions of  $T^*$  are just the reproducing kernels of  $K_\Theta$ . So the completeness of eigenfunctions of  $T^*$  means that any function in  $K_\Theta$  vanishing on  $Z_\varphi$  is zero, that is, that  $Z_\varphi$  is a *uniqueness set* for  $K_\Theta$ .

As a corollary of Theorem 0.7, we will prove that under some mild restrictions any complete and minimal system of reproducing kernels in a model space and its biorthogonal system can be realized (up to a unitary equivalence) as the systems of eigenfunctions of some rank one singular perturbation and of its adjoint (see Theorem 2.5).

Interestingly, completeness problems for a minimal system of reproducing kernels and for its biorthogonal are, in general, not equivalent [8]. This is the reason for existence of certain unexpected examples, as, e.g., in Corollary 0.4. However, for certain classes of perturbations the completeness of  $T^*$  implies the completeness of  $T$ , see Theorems 0.2 and 3.3.

The paper is organized as follows. General properties of rank  $n$  singular perturbations are studied in Section 1, where we give a completeness result in Proposition 1.1, as a corollary of Macaev's theorem. In Section 2 we discuss the model for rank one perturbations and prove Theorem 0.7. In Section 3, sufficient conditions for completeness are given and Theorems 0.1 and 0.2 are proved. In Section 4 we prove our main result about incompleteness, Theorem 0.3. Section 5 contains the counterparts of our results for bounded rank one perturbations of compact operators (e.g., the proofs of Corollary 0.4 and Theorem 0.6). It also contains the proof of Theorem 0.5 and one more result on spectral synthesis. Section 6 (Appendix) contains the proofs of several technical propositions from Section 1.

**0.6. A brief survey of the completeness results.** It should be noted that the literature on completeness of linear operators is very extensive, so here we will give only its very brief overview.

There are several abstract results on completeness we do not mention here, see Dunford and Schwartz's book [22], Part 2, Ch. XI and Part 3, Ch. XIX, §6. In particular, in Theorem XI.9.29, they give a result close to the Keldyš and Macaev's theorems, with no assumption on the triviality of the kernel. We also refer to the books [28], [29] by Gohberg and Krein, in particular, for treatments of the dissipative case and for theorems on the relationship between the sizes of the real and the imaginary part of a compact operator. In the work by G. Gubreev and A. Jerbashian [31], a completeness result, generalizing the Keldyš theorem, is given by applying M. Djrbashian's classes and their factorization theory. See also [10], [11] for an abstract result on similarity of a perturbed operator to a generalized spectral operator and for an application to integro-differential operators with nonlocal boundary conditions. In the book [7], classical results for spaces with indefinite metric are presented. We also mention a more recent monograph [57] by Markus, where also different approaches to the completeness properties of operator pencils are treated with detail. We remark that abstract Banach space results on completeness are also known, see, for instance, [55], [16] and [82].

Much more is known for nonselfadjoint operators corresponding to boundary value problems for ordinary differential equations or systems. The literature devoted to this field is very extensive. We will mention here the works by Malamud (2008), Rykhlov (2009), Shkalikov (1976, 1979, 1982) and Malamud and Oridoroga [54] (2012) (see the references in [54], where an up-to-date account of this work is given), and also the works by Minkin [64] and by Shubov [74]. Reviews [70] and [71] contain a systematic exposition of this field. In [2, §6], abstract results on completeness and their relation to partial differential and pseudodifferential operators on closed manifolds are reviewed.



In some cases when the completeness holds, the Abel summability property for eigenfunction can also be proved, see [50], [60], [47], [26], [80], [13] and the review [2], §6.4.

The completeness of finite-dimensional perturbations of Volterra integral operators and its relationship with expansions by generalized eigenfunctions of ordinary differential operators has been studied by Khromov [43].

Even stronger property of eigenvectors and generalized eigenvectors is to form a Riesz basis or a Riesz basis with parenthesis. There are hundreds of works dedicated to different aspects of this property. In relation with differential operators, these properties are discussed in the above-mentioned reviews, the book [57] and recent works [23], [1], [53], [73] and [27]. Wyss gives in [79] both abstract results and applications to block operator matrices and to differential operators. Notice that the similarity to a normal operator with a discrete spectrum is equivalent to the property of eigenvectors to form a Riesz basis. There are several papers exploiting the model approach, among which we can cite [40], [67] and [76]. In papers [77], [78], certain criteria for the completeness and Riesz basis properties of eigenvectors are obtained in the context of the Naboko's model of nondissipative nonselfadjoint operators.

Recently, the existence of invariant subspaces and other spectral properties of finite rank perturbations of diagonalizable normal operators have been studied in [37], [24] and [25]. We remark that the similarity of a compact perturbation of a normal operator with no eigenvalues to an unperturbed one has also been studied, see [81] and references therein.

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## 1. $d$ -SUBORDINATION OF OPERATORS AND MACAEV TYPE RESULTS ON COMPLETENESS OF RANK $n$ PERTURBATIONS OF SELFADJOINT OPERATORS

First we discuss rank  $n$  perturbations of a compact selfadjoint operator.

**Definition.** Let  $\mathcal{L}_1 \in B(H_1)$ ,  $\mathcal{L}_2 \in B(H_2)$  be two bounded Hilbert space operators. We say that  $\mathcal{L}_2$  is  $d$ -subordinate to  $\mathcal{L}_1$  (and write  $\mathcal{L}_1 \stackrel{d}{\prec} \mathcal{L}_2$ ) if there exists a bounded linear operator  $Y : H_1 \rightarrow H_2$ , which intertwines  $\mathcal{L}_1$  with  $\mathcal{L}_2$  and has a dense range:

$$Y\mathcal{L}_1 = \mathcal{L}_2 Y; \quad \text{clos Ran } Y = H_2.$$

In this situation, if  $k$  is an eigenvector of  $\mathcal{L}_1$ , then  $Yk$  is an eigenvector of  $\mathcal{L}_2$ , and a similar assertion holds for root vectors. It follows that

$$(1.1) \quad \mathcal{L}_1 \stackrel{d}{\prec} \mathcal{L}_2, \mathcal{L}_1 \text{ is complete} \implies \mathcal{L}_2 \text{ is complete.}$$

Let  $\mathcal{A}_0$  be a *bounded*, not necessarily cyclic selfadjoint operator, acting on a Hilbert space. Without loss of generality, we put  $\mathcal{A}_0$  to be the multiplication operator  $M_x$  on a direct integral of Hilbert spaces  $\mathcal{H} \stackrel{\text{def}}{=} \int^\oplus H(x) d\mu(x)$ , where  $\mu$  is a positive measure on  $\mathbb{R}$ . We will assume that  $\ker \mathcal{A}_0 = 0$ , then  $\mu(\{0\}) = 0$ .

Let

$$\mathcal{L}_0 = \mathcal{A}_0 + ab^*, \quad \text{where } a, b : \mathbb{C}^n \rightarrow \mathcal{H},$$

be a rank  $n$  perturbation of  $\mathcal{A}_0$ . Let  $\{e_j\}_{j=1}^n$  be the standard basis in  $\mathbb{C}^n$ . Put  $a_j(x) = (ae_j)(x) \in H(x)$ ,  $|a_j|(x) = \|a_j(x)\|_{H(x)}$  and  $|a|(x) = (\sum_j |a_j(x)|^2)^{1/2}$ . Define  $b_j(x)$ ,  $|b_j|(x)$  and  $|b|(x)$  similarly. Then  $|a|$ ,  $|b|$  are functions in  $L^2(\mu)$ .

We call  $\mathcal{L}_0$  a rank  $n$  generalized weak perturbation of  $\mathcal{A}_0$  if  $\ker \mathcal{L}_0 = 0$  and  $|a| \cdot |b| \in x L^1(\mu)$ . For these perturbations, we define a matrix

$$(1.2) \quad \omega = \omega(\mathcal{A}_0, a, b) = (\omega_{jk})_{j,k=1}^n, \quad \text{where } \omega_{jk} = \int_{\mathbb{R}} x^{-1} \langle a_j(x), b_k(x) \rangle_{H(x)} d\mu(x).$$

The following statement is an easy consequence of Macaev's theorem and of the observation (1.1).

**Proposition 1.1.** *Suppose that  $\mathcal{A}_0 = \mathcal{A}_0^*$  is compact,  $\ker \mathcal{A}_0 = 0$  and  $\mathcal{L}_0$  is a rank  $n$  generalized weak perturbation of  $\mathcal{A}_0$ . If the  $n \times n$  matrix  $I_n + \omega$  is invertible, then  $\mathcal{L}_0$  and  $\mathcal{L}_0^*$  are complete.*

*Proof.* Consider a scalar bounded function

$$\psi(x) = \begin{cases} 1, & \text{if } (|a| \cdot |b|)(x) = 0 \text{ or } |x| \cdot |a|(x)/|b|(x) > 1, \\ \sqrt{|x| \cdot |a|(x)/|b|(x)}, & \text{if } 0 < |x| \cdot |a|(x)/|b|(x) \leq 1, \end{cases}$$

and define the functions  $\tilde{a}_j = a_j/\psi$  and  $\tilde{b}_j = \psi \cdot b_j/x$ . It is easy to check the following facts.

- (1)  $|\tilde{a}_j|, |\tilde{b}_j| \in L^2(\mu)$ ,  $j = 1, \dots, n$ , so that the  $n$ -tuples of functions  $\{\tilde{a}_j\}_1^n, \{\tilde{b}_j\}_1^n$  define operators  $\tilde{a}, \tilde{b} : \mathbb{C}^n \rightarrow \mathcal{H}$ ;
- (2) The bounded operator  $Y = M_\psi$  on  $H$  commutes with  $\mathcal{A}_0 = M_x$  and has a dense range;
- (3)  $Y\tilde{\mathcal{L}}_0 = \mathcal{L}_0 Y$ , where  $\tilde{\mathcal{L}}_0 = (I + \tilde{a}\tilde{b}^*)\mathcal{A}_0$ .

Notice that  $\omega_{jk} = \langle \tilde{a}_j, \tilde{b}_k \rangle_{\mathcal{H}}$ . Since  $I_n + \omega$  is invertible, it follows that  $\ker(I + \tilde{a}\tilde{b}^*) = 0$ . By Macaev's theorem,  $\tilde{\mathcal{L}}_0$  is complete. Since  $\tilde{\mathcal{L}}_0 \stackrel{d}{\prec} \mathcal{L}_0$ ,  $\mathcal{L}_0$  also is complete.  $\square$

Now let us pass to singular perturbations. The next Propositions 1.2–1.6 contain a list of their elementary properties. In these propositions,  $\mathcal{A}$  is assumed to be an arbitrary closed unbounded linear operator, and  $(a, b, \varkappa)$  are  $n$ -data defined in (0.2). The proofs of Propositions 1.2 and 1.3 will be given in the Appendix (Section 6).

In Propositions 1.2–1.6, we make a common assumption that  $0 \notin \sigma(\mathcal{A})$ .

**Proposition 1.2.** (1) *For each  $n$ -data  $(a, b, \varkappa)$ , the operator  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$ , defined in (0.3), is a balanced rank  $n$  singular perturbation of  $\mathcal{A}$ ;*  
 (2) *Any balanced rank  $n$  singular perturbation of  $\mathcal{A}$  has the form  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$ , for some  $n$ -data  $(a, b, \varkappa)$ .*

It is easy to see that, whenever  $\mathcal{L}$  is a finite rank singular perturbation of  $\mathcal{A}$  and  $\sigma(\mathcal{A}) \cup \sigma(\mathcal{L}) \neq \mathbb{C}$ ,  $\mathcal{L}$  is a balanced perturbation of  $\mathcal{A}$ .

Suppose  $n$ -data  $(a, b, \varkappa)$  are fixed. We recall that the adjoint of a closed operator on a Hilbert space exists if and only if this operator is densely defined. To be able to consider  $\mathcal{L}^*$ , we need to introduce a dual condition to  $(A_n)$ : for any  $d \in \mathbb{C}^n$ ,

$$(\mathcal{A}^*)^{-1}bd \in \mathcal{D}(\mathcal{A}^*), \quad \varkappa^*d = a^*((\mathcal{A}^*)^{-1}bd) \implies d = 0. \quad (A_n^*)$$

**Proposition 1.3.** (1) *Operator  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is densely defined if and only if the data  $(a, b, \varkappa)$  satisfy  $(A_n^*)$ .*  
 (2) *If  $(A_n^*)$  holds, then  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)^* = \mathcal{L}(\mathcal{A}^*, b, a, \varkappa^*)$ .*

The next proposition is a kind of a uniqueness assertion.

**Proposition 1.4.** (1) *For any invertible operators  $\tau_1, \tau_2$  on  $\mathbb{C}^n$ ,  $\mathcal{L}(\mathcal{A}, a, b, \varkappa) = \mathcal{L}(\mathcal{A}, a\tau_1^{-1}, b\tau_2, \tau_2^*\varkappa\tau_1^{-1})$ ;*

- (2) Conversely, if  $a, b, a_1, b_1 : \mathbb{C}^n \rightarrow \mathcal{A}H$  are rank  $n$  operators and  $\mathcal{L}(\mathcal{A}, a, b, \varkappa) = \mathcal{L}(\mathcal{A}, a_1, b_1, \varkappa_1)$ , then there are invertible operators  $\tau_1, \tau_2$  on  $\mathbb{C}^n$  such that  $a_1 = a\tau_1^{-1}$ ,  $b_1 = b\tau_2$ ,  $\varkappa_1 = \tau_2^* \varkappa \tau_1^{-1}$ .

**Proposition 1.5.** (1) Given operators  $a, b$  of rank  $n$  and an invertible  $\varkappa$ , the operator  $\mathcal{A}^{-1} - (\mathcal{A}^{-1}a)\varkappa^{-1}(b^*\mathcal{A}^{-1})$  has a trivial kernel (and therefore has an inverse in the algebraic sense, defined on its image) if and only if the triple  $(a, b, \varkappa)$  satisfies  $(A_n)$ .  
 (2) If the triple  $(a, b, \varkappa)$  satisfies  $(A_n)$  and  $\varkappa$  is invertible, then

$$\mathcal{L}(\mathcal{A}, a, b, \varkappa) = (\mathcal{A}^{-1} - (\mathcal{A}^{-1}a)\varkappa^{-1}(b^*\mathcal{A}^{-1}))^{-1}.$$

Therefore, in this case  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is an (unbounded) algebraical inverse to a bounded rank  $n$  perturbation of the bounded operator  $\mathcal{A}^{-1}$ .

**Proposition 1.6.** For any  $\lambda \in \rho(\mathcal{A})$ ,

$$\mathcal{L}(\mathcal{A}, a, b, \varkappa) - \lambda I = \mathcal{L}(\mathcal{A} - \lambda I, a, b, \varkappa(\lambda)),$$

where  $\varkappa(\lambda) = \varkappa + \lambda b^*(\mathcal{A} - \lambda)^{-1}\mathcal{A}^{-1}a$ .

The proofs of Propositions 1.4 – 1.6 are straightforward, and we omit them.

We call  $\mathcal{L}$  a rank  $n$  generalized weak singular perturbation of  $\mathcal{A}$  if  $0 \notin \sigma(\mathcal{A})$  and  $|a| \cdot |b|/x \in L^1(\mu)$ . For these perturbations, we will also use (1.2) to define an  $n \times n$  matrix  $\omega(\mathcal{A}, a, b)$ . We remark that if  $\varkappa - \omega(\mathcal{A}, a, b)$  is invertible, then conditions  $(A_n)$  and  $(A_n^*)$  hold, which implies that  $\mathcal{L}$  and  $\mathcal{L}^*$  are correctly defined.

**Proposition 1.7.** Suppose that  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is a rank  $n$  generalized weak singular perturbation of a selfadjoint operator  $\mathcal{A}$  with compact resolvent. If the  $n \times n$  matrix  $\varkappa - \tilde{\omega}$  is invertible, then  $\mathcal{L}^*$  is correctly defined, and  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.

*Proof.* Let us apply first Proposition 1.6. The Lebesgue dominated convergence theorem implies that  $\varkappa(iy) \rightarrow \varkappa - \tilde{\omega}$  as  $y \rightarrow +\infty$ . Hence for all but a discrete set of  $\lambda$ 's in  $\mathbb{C}^+$ ,  $\varkappa(\lambda)$  is invertible and therefore the same holds true for all  $\lambda$ 's on the real line, except a discrete set in  $\mathbb{R} \setminus \sigma(\mathcal{A})$  (recall that  $\sigma(\mathcal{A})$  is discrete).

By substituting  $\mathcal{A}$  with  $\mathcal{A} - \lambda I$  for some real  $\lambda$ , if necessary, we can assume that  $\varkappa(0) = \varkappa$  is invertible. By Proposition 1.5,  $\mathcal{L}^{-1} = \mathcal{A}^{-1} + \widehat{a}\widehat{b}^*$ , where  $\widehat{a} = -\mathcal{A}^{-1}a\varkappa^{-1}$  and  $\widehat{b} = \mathcal{A}^{-1}b$ . Put  $\tilde{\omega} = \omega(\mathcal{A}, a, b)$ . Then it is easy to check that  $\omega(\mathcal{A}^{-1}, \widehat{a}, \widehat{b}) = -\tilde{\omega}\varkappa^{-1}$ . Hence the  $n \times n$  matrix  $\omega(\mathcal{A}^{-1}, \widehat{a}, \widehat{b}) + I_n = (\varkappa - \tilde{\omega})\varkappa^{-1}$  is invertible. By Proposition 1.1,  $\mathcal{L}^{-1}$  and  $(\mathcal{L}^*)^{-1}$  are complete. Hence  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.  $\square$

## 2. BASIC FUNCTIONAL MODEL IN $K_\Theta$

If  $f, g$  are vectors in a Hilbert space  $H$ , we denote by  $g^*$  the linear functional  $g^*x \stackrel{\text{def}}{=} \langle x, g \rangle$  on  $H$  and by  $fg^*$  the rank one linear operator on  $H$ , given by  $(fg^*)x \stackrel{\text{def}}{=} (g^*x)f$ . Here we restrict ourselves to a discussion of a singular rank one perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  of a cyclic selfadjoint operator  $\mathcal{A}$ , given by (0.4), where  $\mu$  is a singular measure. Notice first of all that  $\mathcal{L}$  can be expressed as follows:

$$(2.1) \quad \mathcal{L}y = (\mathcal{A} + a(b^*)_{a,\varkappa})y \stackrel{\text{def}}{=} \mathcal{A}y + ((b^*)_{a,\varkappa}y)a, \quad y \in \mathcal{D}(\mathcal{L}),$$

where the linear functional

$$(b^*)_{a,\varkappa} : \mathcal{D}(\mathcal{L}) \rightarrow \mathbb{C}$$

is defined by

$$(2.2) \quad (b^*)_{a,\varkappa} y \stackrel{\text{def}}{=} -c$$

whenever  $y = y_0 + c \cdot \mathcal{A}^{-1}a$  is a decomposition of a vector  $y \in \mathcal{D}(\mathcal{L})$  as in (0.6). The summands in the right hand part of (2.1) are elements of  $xL^2(\mu)$  and in general do not belong to  $L^2(\mu)$ .

If  $b \in L^2(\mu)$ , then  $(b^*)_{a,\varkappa}$  is a bounded functional, given by

$$(b^*)_{a,\varkappa} y = \frac{1}{\varkappa - \langle \mathcal{A}^{-1}a, b \rangle} \langle y, b \rangle, \quad y \in L^2(\mu).$$

If, moreover,  $a, b \in L^2(\mu)$ , then  $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$ , and  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is just a bounded rank one perturbation of  $\mathcal{A}$ . In particular,

$$\mathcal{L}(\mathcal{A}, a, b, \varkappa) = \mathcal{A} + ab^*, \quad \text{if } \varkappa = 1 + \langle \mathcal{A}^{-1}a, b \rangle.$$

Other values of  $\varkappa$  do not enlarge the scope of perturbations considered. Therefore in the case when  $a, b \in L^2(\mu)$ , we may consider the usual rank one perturbation

$$\mathcal{L}(\mathcal{A}, a, b) \stackrel{\text{def}}{=} \mathcal{A} + ab^*.$$

However, most of the time we do not need to distinguish this case and all the results remain valid in this situation.

**2.1. Model spaces and Clark measures.** We recall that the functions  $\beta, \rho, \Theta$  and  $\varphi$  have been defined above by formulas (0.8)–(0.11). Since  $\mu(\{0\}) = 0$ ,  $zb^*(\mathcal{A} - z)^{-1}\mathcal{A}^{-1}a \neq \text{const}$ , and therefore  $\beta \not\equiv 0$  in  $\mathbb{C} \setminus \mathbb{R}$ .

Here we will discuss the model space  $K_\Theta$ , the so-called Clark measures, related to it, and Clark orthogonal bases of reproducing kernels in  $K_\Theta$ . If we identify the functions in  $H^2$  with their boundary values on  $\mathbb{R}$ , then an equivalent definition of  $K_\Theta$  is  $K_\Theta = H^2 \cap \Theta \overline{H^2}$ . Thus, we have a criterion for the inclusion  $f \in K_\Theta$  which we will repeatedly use:

$$(2.3) \quad f \in K_\Theta \iff f \in H^2 \text{ and } \Theta \overline{f} \in H^2.$$

In other words, if  $f \in K_\Theta$ , then the function  $\tilde{f}(x) = \Theta(x)\overline{f(x)}$ ,  $x \in \mathbb{R}$ , is also in  $K_\Theta$  (in the sense that it coincides with nontangential boundary values of a function from  $K_\Theta$ ). If  $\Theta$  is meromorphic, then any  $f \in K_\Theta$  is also meromorphic, and the formula  $\tilde{f}(z) = \Theta(z)\overline{f(\bar{z})}$  holds for all  $z \in \mathbb{C}^+$ .

The following statement is an immediate corollary of (2.3).

**Lemma 2.1.** *Let  $\Theta$  be inner and let  $\frac{\varphi}{z+i} \in H^2$ . Assume that  $\varphi$  satisfies either (0.12) or (0.13). Then*

- (1) *If  $\varphi \in H^2$ , then  $\varphi \in K_\Theta$ .*
- (2) *Let  $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$  and let  $\varphi$  be analytic in a neighborhood of  $\lambda$ . Then the function  $g_\lambda(z) = \frac{\varphi(z) - \varphi(\lambda)}{z - \lambda}$  belongs to  $K_\Theta$ .*

*Proof.* We either have

$$\frac{\Theta(x)\overline{\varphi(x)}}{x+i} = \Theta(x)\overline{\psi(x)} + \frac{\overline{\varphi(i)}\Theta(x)}{x+i}$$

for some  $\psi \in K_\Theta$ , or  $\Theta(x)\overline{\varphi(x)} = \varphi(x)$  a.e. on  $\mathbb{R}$ . In any of these cases,  $\frac{\Theta\overline{\varphi}}{x+i} \in H^2$ . If, moreover,  $\varphi \in H^2$ , then  $\varphi \in K_\Theta$  by (2.3), which proves (1).

(2) Obviously,  $g_\lambda \in H^2$ . Also,  $\Theta(x)\overline{g_\lambda(x)}(x - \bar{\lambda}) = \Theta(x)\overline{\varphi(x)} - \overline{\varphi(\lambda)}\Theta(x)$ , whence  $\Theta\overline{g_\lambda} \in H^2$ .  $\square$

For  $\lambda \in \mathbb{C}^+$  set

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{z - \bar{\lambda}}, \quad \tilde{k}_\lambda(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}.$$

Note that  $\tilde{k}_\lambda(z) = \Theta(z)\overline{k_\lambda(\bar{z})}$ . The definitions of  $k_\lambda$  and  $\tilde{k}_\lambda$  can be extended to the points  $\lambda \in \mathbb{R}$  such that  $\Theta$  has an analytic extension to a neighborhood of  $\lambda$ , and  $\tilde{k}_\lambda = -\Theta(\lambda)k_\lambda$  for these values of  $\lambda \in \mathbb{R}$ .

Note that  $k_\lambda$  is the *reproducing kernel* of  $K_\Theta$  corresponding to the point  $\lambda$ , i.e.,

$$(2.4) \quad \langle f, k_\lambda \rangle = 2\pi i f(\lambda), \quad f \in K_\Theta.$$

Analogously,  $\langle f, \tilde{k}_\lambda \rangle = -2\pi i \overline{f(\lambda)}$ .

Orthogonal bases of reproducing kernels were studied by L. de Branges [14] for meromorphic inner functions and by D.N. Clark [18] in the general case. They may be constructed as follows. For any  $\zeta \in \mathbb{C}$ ,  $|\zeta| = 1$ , the function  $(\zeta + \Theta)/(\zeta - \Theta)$  has positive real part in the upper half-plane. Then, by the Herglotz theorem, there exist  $p_\zeta \geq 0$ ,  $q_\zeta \in \mathbb{R}$  and a measure  $\sigma_\zeta$  with  $\int_{\mathbb{R}} (1+t^2)^{-1} d\sigma_\zeta(t) < \infty$  such that

$$(2.5) \quad \frac{\zeta + \Theta(z)}{\zeta - \Theta(z)} = -ip_\zeta z + iq_\zeta + \frac{1}{i} \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma_\zeta(t), \quad z \in \mathbb{C}^+.$$

We will say that  $\sigma_\zeta$  is a *Clark measure* of  $\Theta$ . It follows from the results of Clark [18] (for the setting of the unit disc, instead of the upper half-plane) that in the case when  $p_\zeta = 0$  the mapping

$$(2.6) \quad (U_\zeta f)(z) = \sqrt{\pi}(\zeta - \Theta(z)) \int_{\mathbb{R}} \frac{f(t)}{t-z} d\sigma_\zeta(t)$$

is a unitary operator from  $L^2(d\sigma_\zeta)$  onto  $K_\Theta$  (see Proposition 2.3 below). The inverse operator to  $U_\zeta$  has the sense just of the embedding  $K_\Theta \subset L^2(\pi\sigma_\zeta)$ ; clearly, it also is unitary. As Poltoratskii has shown in [68], any function  $f \in K_\Theta$  has finite nontangential boundary values  $\sigma_\zeta$ -a.e.

In particular, if  $\sigma_\zeta$  is purely atomic (that is,  $\sigma_\zeta = \sum_n c_n \delta_{t_n}$ , where  $\delta_x$  denotes the Dirac measure at the point  $x$ ) and  $p_\zeta = 0$ , then  $k_{t_n} \in K_\Theta$  and the system  $\{k_{t_n}\}$  of reproducing kernels is an orthogonal basis in  $K_\Theta$ . Of course, if  $\Theta$  is meromorphic, then any Clark measure  $\sigma_\zeta$  is atomic.

If  $p_\zeta > 0$  in the representation (2.5), then the orthogonal complement to the span of  $\{k_{t_n}\}$  is one-dimensional and is generated by the function  $\zeta - \Theta$ , which in this case belongs to  $K_\Theta$  (this is half-plane version of a result due to Ahern and Clark [3]). Thus,

$$(2.7) \quad \zeta - \Theta \in H^2 \iff \zeta - \Theta(iy) = O(y^{-1}), \quad y \rightarrow \infty \iff p_\zeta > 0.$$

We will use the notation

$$(2.8) \quad \nu = |b|^2 \mu.$$

Returning to our model of singular perturbations, observe that the representations (0.9)–(0.10) mean that  $\nu$  is the Clark measure  $\sigma_{-1}$  for  $\Theta$ . By the results of [68],

$$(2.9) \quad \varphi = ia/b \quad \nu\text{-a.e.}$$

**Proposition 2.2.** *Let  $a, b$  be functions that satisfy (0.5) and let  $\varkappa \in \mathbb{R}$ . Let  $\Theta$  and  $\varphi$  be defined by (0.10) and (0.11). Then we have:*

- (1)  $1 + \Theta \notin H^2$  and  $\Theta(0) \neq -1$ ;
- (2)  $\frac{\varphi(z) - \varphi(i)}{z - i} \in K_\Theta$ ;
- (3) If  $a \notin L^2(\mu)$ , then  $\varphi \notin H^2$ ;



- (4) If  $a \in L^2(\mu)$ , then  $\varphi \in H^2$  if and only if  $\varkappa = \int x^{-1}a(x)\overline{b(x)}d\mu(x)$ ;  
 (5)  $\varkappa = 0$  if and only if  $\varphi(0) = 0$ .

*Proof.* (1) It follows from (0.9) that  $|\rho(iy)| = o(y)$  as  $y \rightarrow +\infty$ . Since  $\frac{1-\Theta}{1+\Theta} = -i\rho$ , it follows from (2.7) that  $1 + \Theta \notin H^2$ .

(2) It follows from the formula (0.11) for  $\varphi$  that

$$\frac{\varphi(z) - \varphi(i)}{z - i} = \frac{1 + \Theta(z)}{2} \int \frac{a(x)\overline{b(x)}}{(x - z)(x - i)} d\mu(x) + \beta(i) \frac{\Theta(z) - \Theta(i)}{2(z - i)}.$$

Since  $\frac{a}{x-i} \in L^2(\mu)$ , we have  $\frac{a}{(x-i)b} \in L^2(\nu)$ , where  $\nu = |b|^2\mu$ . Since  $\nu$  is the Clark measure  $\sigma_{-1}$  for  $K_\Theta$ , and the Clark operator  $U_{-1}$  maps  $L^2(\nu)$  onto  $K_\Theta$ , we have  $\frac{\varphi(z) - \varphi(i)}{z - i} \in K_\Theta$ .

(3) If  $\varphi \in H^2$ , then  $\varphi \in K_\Theta$ . Hence,  $\varphi \in L^2(\nu)$ , and, since  $\varphi = ia/b$   $\nu$ -a.e., we have  $\int |a(x)|^2 d\mu(x) = \int |\varphi(x)|^2 d\nu(x) < \infty$ .

(4) Now let  $a \in L^2(\mu)$ . Then we have

$$2\varphi(z) = (1 + \Theta(z)) \left( \varkappa - \int \frac{a(x)\overline{b(x)}}{x} d\mu(x) \right) + (1 + \Theta(z)) \int \frac{a(x)\overline{b(x)}}{x - z} d\mu(x).$$

Since  $\int |a|^2 |b|^{-2} d\nu = \int |a|^2 d\mu < \infty$ , the boundedness of the operator  $U_{-1}$  implies that the last term is in  $K_\Theta$ . Hence,  $\varphi \in K_\Theta$  if and only if  $(1 + \Theta(z)) \left( \varkappa - \int x^{-1}a(x)\overline{b(x)}d\mu(x) \right)$  is in  $K_\Theta$ . Since  $1 + \Theta \notin K_\Theta$ , we conclude that  $\varphi \in K_\Theta$  if and only if the coefficient is zero.

(5) Obviously,  $\varkappa = 0$  if and only if  $\beta(0) = 0$ . By (1), we have  $1 + \Theta(0) \neq 0$ , whence the statement follows.  $\square$

**2.2. Proof of Theorem 0.7 on the model in  $K_\Theta$ .** We define “diagonalizing” transforms

$$(2.10) \quad V_0 u(z) = 2\sqrt{\pi} b^*(\mathcal{A} - z)^{-1} u, \quad z \in \rho(\mathcal{A}), u \in L^2(\mu),$$

$$(2.11) \quad V_{\mathcal{L}} u(z) = 2\sqrt{\pi} (b^*)_{a,\varkappa} (\mathcal{L} - z)^{-1} u, \quad z \in \rho(\mathcal{L}), u \in L^2(\mu).$$

First we need the following proposition.

**Proposition 2.3.** *Let  $\Theta$  and  $\varphi$  be defined by (0.10) and (0.11). Then*

- (1)  $V_0$  is an isometric isomorphism of  $L^2(\mu)$  onto  $\frac{1}{1+\Theta}K_\Theta$ ;  
 (2)  $V_{\mathcal{L}}$  is an isometric isomorphism of  $L^2(\mu)$  onto  $\frac{1}{\varphi}K_\Theta$ .

*Proof.* First let us deduce the splitting formula

$$(2.12) \quad V_{\mathcal{L}} u(z) = \beta(z)^{-1} V_0 u(z), \quad u \in L^2(\mu).$$

To do that, choose any  $u \in L^2(\mu)$  and put  $y \stackrel{\text{def}}{=} (\mathcal{L} - z)^{-1} u \in \mathcal{D}(\mathcal{L})$ . A direct calculation shows that the representation  $y = y_0 + c\mathcal{A}^{-1}a$  as in (0.6) is given by

$$y_0 = (\mathcal{A} - z)^{-1} u + cz(\mathcal{A} - z)^{-1} \mathcal{A}^{-1} a, \quad c = -\beta(z)^{-1} V_0 u(z).$$

By (2.2), this implies (2.12).

Statement (1) is very close to the results by Clark [18]. To prove it, one can apply the arguments given in [17, Proposition 9.5.4]. Namely, for  $\xi \notin \text{supp } \mu$ , put

$$\eta_\xi(x) = \frac{1}{\rho(\xi) + i} \frac{b(x)}{x - \xi} \in L^2(\mu).$$

Let  $\xi, \tau \notin \text{supp } \mu \cup \bar{Z}_\Theta$ . Direct calculations give that

$$(1 + \Theta(z))V_0\eta_\xi(z) = -\frac{\Theta(z) - \Theta(\xi)}{2\sqrt{\pi}(z - \xi)} = \frac{\Theta(\xi)}{2\sqrt{\pi}}k_{\bar{\xi}}(z) \in K_\Theta;$$

$$\langle \eta_\xi, \eta_\tau \rangle_{L^2(\mu)} = -\frac{1}{2i} \frac{1 - \overline{\Theta(\tau)}\Theta(\xi)}{\xi - \bar{\tau}} = \left\langle \frac{\Theta(\xi)}{2\sqrt{\pi}}k_{\bar{\xi}}, \frac{\Theta(\tau)}{2\sqrt{\pi}}k_{\bar{\tau}} \right\rangle_{K_\Theta}.$$

Since  $\{\eta_\xi\}$  are complete in  $L^2(\mu)$  and  $\{\Theta(\bar{\xi})k_{\bar{\xi}}\}$  are complete in  $K_\Theta$ , the assertion (1) follows.

Statement (2) follows from statement (1) and formula (2.12).  $\square$

*Proof of Theorem 0.7.* Put  $V_{\mathcal{L}, \varphi} u \stackrel{\text{def}}{=} \varphi \cdot V_{\mathcal{L}} u$ ,  $u \in L^2(\mu)$ . By Proposition 2.3,  $V_{\mathcal{L}, \varphi}$  is an isometric isomorphism from  $L^2(\mu)$  onto  $K_\Theta$ . Define an operator  $T = V_{\mathcal{L}, \varphi} \mathcal{L} V_{\mathcal{L}, \varphi}^{-1}$  on  $K_\Theta$ . It is unitarily equivalent to  $\mathcal{L}$ . The splitting formula

$$V_{\mathcal{L}}(\mathcal{L} - \xi)^{-1}u(z) = \frac{V_{\mathcal{L}}u(z) - V_{\mathcal{L}}u(\xi)}{z - \xi}, \quad \xi \notin \sigma(\mathcal{L}),$$

is immediate. It easily implies the expressions for  $\mathcal{D}(T)$  and for the action of  $T$ , given in the Theorem.

The properties of  $\Theta$  and  $\varphi$  follow immediately from Proposition 2.2. If, moreover,  $\mathcal{L}$  is a real type perturbation, then  $\beta$  is real a.e. on  $\mathbb{R}$ , and it follows that  $\Theta$  and  $\varphi$  satisfy (0.13).

We turn to the proof of the converse statement and show that any pair  $(\Theta, \varphi)$  with the above properties can be realized in our model. Let  $\nu$  be the Clark measure  $\sigma_{-1}$  for  $\Theta$ . Note that by the hypothesis  $0 \notin \text{supp } \nu$  and also that, by (2.7),  $p_{-1} = 0$ . Since  $\frac{\varphi(z) - \varphi(i)}{z - i}$  belongs to the space  $K_\Theta$ , by (2.6) there exists  $u \in L^2(\nu)$  such that

$$\frac{\varphi(z) - \varphi(i)}{z - i} = (1 + \Theta(z)) \int \frac{u(t)}{t - z} d\nu(t).$$

Choose any  $b$  so that  $|b| > 0$   $\nu$ -a.e. and put  $\mu = |b|^{-2}\nu$ . Then  $x^{-1}b \in L^2(\mu)$ . We have

$$\varphi(z) = \varphi(i) + (1 + \Theta(z)) \left[ \int \left( \frac{1}{x - z} - \frac{1}{x} \right) (x - i)u(x)|b(x)|^2 d\mu(x) - \int \frac{u(x)}{x} d\nu(x) \right].$$

Since, by definition of the Clark measure (2.5),

$$(1 + \Theta(z)) \left( r_0 + \frac{1}{\pi i} \int \left( \frac{1}{x - z} - \frac{1}{x} \right) d\nu(x) \right) = 2$$

for some constant  $r_0$ , we can write  $\varphi(i)$  as an analogous integral and finally obtain

$$\varphi(z) = \frac{1 + \Theta(z)}{2} \left( \varkappa + \int \left( \frac{1}{x - z} - \frac{1}{x} \right) v(x)|b(x)|^2 d\mu(x) \right),$$

for some constant  $\varkappa$  and  $v$  such that  $x^{-1}v \in L^2(\nu) = L^2(|b|^2\mu)$ . Put  $a(x) = v(x)/b(x)$ . Then  $x^{-1}a \in L^2(\mu)$  and

$$\varphi(z) = \frac{1 + \Theta(z)}{2} \left( \varkappa + \int \left( \frac{1}{x - z} - \frac{1}{x} \right) a(x)\bar{b}(x) d\mu(x) \right).$$

Since  $\varphi \notin K_\Theta$ , by Proposition 2.2, either  $a \notin L^2(\mu)$  or  $\varkappa \neq \int x^{-1}a(x)\bar{b}(x) d\mu(x)$  and so (A) is satisfied. We conclude that  $\Theta$  and  $\varphi$  correspond to the singular perturbation associated with the measure  $\mu$  and the data  $a$ ,  $b$  and  $\varkappa$ .

Finally, note that if  $\Theta$  and  $\varphi$  satisfy (0.13), then  $\beta$  is real on  $\mathbb{R}$  whence  $a\bar{b} \in \mathbb{R}$  and  $\varkappa \in \mathbb{R}$ . Thus the constructed perturbation is of real type.  $\square$

**Remark.** It follows from this theorem that if two operators  $\mathcal{L}$ ,  $\mathcal{L}_1$  as above are real type perturbations and  $\varphi = \varphi_1$ , then  $\mathcal{L}$  and  $\mathcal{L}_1$  are unitarily equivalent.

**2.3. The spectrum and eigenfunctions of  $T$  and of  $T^*$ .** Throughout this section we will assume that  $\Theta$  and  $\varphi$  are meromorphic. Then it follows from Lemma 2.1 that for any  $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$  such that  $\varphi(\lambda) = 0$ , the function

$$h_\lambda(z) = \frac{\varphi(z)}{z - \lambda}$$

belongs to  $K_\Theta$ . Denote by  $Z_\varphi$  the set of zeros of  $\varphi$  in  $\mathbb{C}^+ \cup \mathbb{R}$ . Recall that if we put  $\tilde{\varphi}(z) = \Theta(z)\overline{\varphi(\bar{z})}$ , then  $\tilde{\varphi}$  is analytic in  $\mathbb{C}^+$ . Denote by  $Z_{\tilde{\varphi}}$  the zero set of  $\tilde{\varphi}$  in  $\mathbb{C}^+ \cup \mathbb{R}$  (note that the zeros of  $\varphi$  and  $\tilde{\varphi}$  on  $\mathbb{R}$  coincide).

The following lemma describes the spectrum of the model operator  $T$  and of its adjoint. For a set  $Z \subset \mathbb{C}$  we denote by  $\bar{Z}$  the set  $\{\bar{z} : z \in Z\}$ .

**Lemma 2.4.** *Let meromorphic  $\Theta$  and  $\varphi$  correspond to a singular rank one perturbation of a cyclic selfadjoint operator  $\mathcal{A}$  with the compact resolvent. Then the following holds:*

- (1) *Operators  $\mathcal{L}$  and  $T$  have compact resolvents;*
- (2)  *$\sigma(T) = \sigma_p(T) = Z_\varphi \cup \bar{Z}_{\tilde{\varphi}}$ ;*
- (3) *The eigenspace of  $T$  corresponding to an eigenvalue  $\lambda$ ,  $\lambda \in Z_\varphi \cup \bar{Z}_{\tilde{\varphi}}$ , is spanned by  $h_\lambda$ ;*
- (4) *Suppose that either  $\text{Im } \lambda \geq 0$  and  $\lambda$  is a zero of  $\varphi$  (exactly) of order  $k$ , or  $\text{Im } \lambda < 0$  and  $\bar{\lambda}$  is a zero of  $\tilde{\varphi}$  of order  $k$ . Then  $\dim \ker(T - \lambda)^\ell = \ell$  for  $\ell \leq k$ , and  $\ker(T - \lambda)^s = \ker(T - \lambda)^k$  for  $s > k$ . Moreover, for  $1 \leq \ell \leq k$ , the space  $\ker(T - \lambda)^\ell$  is spanned by the eigenvectors and root vectors  $\varphi(z)/(z - \lambda)$ ,  $\varphi(z)/(z - \lambda)^2$ ,  $\dots$ ,  $\varphi(z)/(z - \lambda)^\ell$  of  $T$ .*
- (5) *Suppose  $T^*$  is correctly defined. Then  $\sigma(T^*) = \bar{Z}_\varphi \cup Z_{\tilde{\varphi}}$  and  $\ker(T^* - \bar{\lambda}I)$  is spanned by  $k_\lambda$  for  $\lambda \in Z_\varphi$ , while  $\ker(T^* - \lambda I)$  is spanned by  $\tilde{k}_\lambda$  for  $\lambda \in Z_{\tilde{\varphi}}$ .*

*Proof.* Since  $\mathcal{A}$  has a compact resolvent and  $\mathcal{L}$  is its finite rank perturbation in the sense of [6],  $(\mathcal{L} - \lambda I)^{-1}$  is compact for any  $\lambda \notin \sigma(\mathcal{L})$ . Hence the resolvent of  $T$  also is compact. This gives statement (1) and also implies that  $\sigma(T) = \sigma_p(T)$ .

Now let us describe the eigenvalues and eigenfunctions of  $T$ . If for some  $\lambda \in \mathbb{C}$

$$Tf = zf - c\varphi = \lambda f,$$

then  $f = \frac{c\varphi}{z - \lambda}$ . Hence  $\lambda$  is in  $\sigma_p(T)$  if and only if  $\frac{\varphi(z)}{z - \lambda} \in K_\Theta$ . It follows also that for any  $\lambda \in \sigma_p(T)$ , the eigenspace  $\ker(T - \lambda)$  is one-dimensional. If  $\text{Im } \lambda \geq 0$ ,  $\frac{\varphi(z)}{z - \lambda} \in K_\Theta$  if and only if  $\varphi(\lambda) = 0$  (see Lemma 2.1). If  $\text{Im } \lambda < 0$ , then the inclusion  $\frac{\varphi(z)}{z - \lambda} \in K_\Theta$  is equivalent to  $\frac{\Theta(z)\overline{\varphi(\bar{z})}}{z - \lambda} = \frac{\tilde{\varphi}(z)}{z - \lambda} \in H^2(\mathbb{C}^+)$ , which happens if and only if  $\tilde{\varphi}(\bar{\lambda}) = 0$ . This proves statements (2) and (3).

It is easy to check statement (4) by applying induction in  $\ell$ ; we omit the details.

Now suppose that  $T$  has an adjoint  $T^*$ . The above observations imply that  $\sigma_p(T^*) = \bar{Z}_\varphi \cup Z_{\tilde{\varphi}}$ , and that all eigenspaces of  $T^*$  are one-dimensional. Now let  $\lambda \in Z_\varphi$ . Then

$$\langle f, T^*k_\lambda \rangle = \langle Tf, k_\lambda \rangle = (zf(z) - c\varphi(z))|_{z=\lambda} = \lambda f(\lambda) = \lambda \langle f, k_\lambda \rangle$$

for any  $f \in \mathcal{D}(T)$ . Since  $\mathcal{D}(T)$  is dense in  $K_\Theta$ , one has  $T^*k_\lambda = \bar{\lambda}k_\lambda$ . Analogously, using the equality  $\langle f, \tilde{k}_\lambda \rangle = -2\pi i \tilde{f}(\lambda)$ , it is easy to show that  $T^*\tilde{k}_\lambda = \lambda \tilde{k}_\lambda$  for  $\lambda \in Z_{\tilde{\varphi}}$ . This gives statement (5).  $\square$

**Remarks.** 1. If  $\mathcal{A}$  is cyclic and compact, then  $\mathcal{L}$  also is compact, and  $\varphi$  is meromorphic in  $\mathbb{C} \setminus \{0\}$ . Items (3)–(5) of the above Lemma apply to any  $\lambda \neq 0$ . It follows, in particular, that in this case,  $\mathcal{L}$  is a Volterra operator (that is,  $\sigma(\mathcal{L}) = \{0\}$ ) if and only if  $\varphi(\lambda) \neq 0$  for all  $\lambda \neq 0$  with  $\text{Im } \lambda \geq 0$ .

2. It is clear that the system  $\{k_\lambda\}_{\lambda \in \sigma_p(T^*)}$  is (up to normalization) biorthogonal to the system  $\{h_\lambda\}_{\lambda \in \sigma_p(T)}$ .

3. A statement analogous to Lemma 2.4 holds for general model spaces (not necessarily associated with meromorphic inner functions). E.g., if  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  or  $\lambda \in \mathbb{R}$  and  $\Theta$  is analytic in a neighborhood of  $\lambda$ , we have  $\lambda \in \sigma(T)$  if and only if  $\lambda \in \sigma_p(T)$  and if and only if  $\varphi(\lambda) = 0$ . In this case,  $h_\lambda$  is an eigenfunction of  $T$  while  $k_\lambda$  is an eigenfunction of  $T^*$  (if  $T^*$  is correctly defined).

As a corollary of Theorem 0.7 we show that under certain restrictions complete and minimal systems of reproducing kernels in a space  $K_\Theta$  and their biorthogonal systems can be realized as systems of eigenfunctions of an operator which is unitarily equivalent to some singular rank one perturbation of a selfadjoint operator.

**Theorem 2.5.** *Let  $\Theta$  be an inner function analytic in a neighborhood of 0 and such that  $1 + \Theta \notin K_\Theta$ . Let the system of reproducing kernels  $\{k_\lambda\}_{\lambda \in \Lambda}$ ,  $\Lambda \subset \mathbb{C}^+$  (or  $\Lambda \subset \text{clos } \mathbb{C}^+$  in the case when  $\Theta$  is meromorphic), be complete and minimal in  $K_\Theta$ . Then there exists a function  $\varphi$  such that  $\varphi \notin H^2$ ,  $\varphi$  vanishes exactly on the set  $\Lambda$  and  $\frac{\varphi(z)}{z-\lambda} \in K_\Theta$  for any  $\lambda \in \Lambda$ . Moreover,  $\varphi$  is determined uniquely up to a multiplicative constant, and the following statements hold.*

- (1)  $\Theta$  and  $\varphi$  correspond to some singular rank one perturbation  $\mathcal{L}$  of the multiplication operator in  $L^2(\mu)$  with  $x^{-1}a, x^{-1}b \in L^2(\mu)$ .
- (2) If, moreover,  $\zeta - \Theta \notin H^2$  for any  $\zeta \in \mathbb{C}$ ,  $|\zeta| = 1$ , then the adjoint  $\mathcal{L}^*$  of  $\mathcal{L}$  exists and the system  $\{k_\lambda\}_{\lambda \in \Lambda}$  is the set of eigenfunctions of the corresponding operator  $T^*$ .
- (3) If, moreover,  $\Theta$  is a meromorphic inner function and  $\Lambda \subset \mathbb{R}$ , then there is a constant  $\xi$ ,  $|\xi| = 1$ , such that  $\Theta$  and  $\xi\varphi$  correspond to some almost Hermitian singular rank one perturbation.

If  $\varphi$  is the function, defined as above, we will refer to it as a *generating function* for the system  $\{k_\lambda\}_{\lambda \in \Lambda}$ .

*Proof.* (1) Since the system  $\{k_\lambda\}_{\lambda \in \Lambda}$  is complete and minimal, for a fixed  $\lambda_0 \in \Lambda$  there exists a unique (up to a constant factor) function  $g \in K_\Theta$  such that  $g(\lambda) = 0$ ,  $\lambda \in \Lambda \setminus \{\lambda_0\}$ . Put  $\varphi = (z - \lambda_0)g$ . Then  $\varphi$  vanishes exactly on the set  $\Lambda$ . Clearly,  $\varphi \notin H^2$  and  $\frac{\varphi(z)}{z-\lambda} \in K_\Theta$  for any  $\lambda \in \Lambda$ . Thus, by the converse statement in Theorem 0.7,  $\Theta$  and  $\varphi$  correspond to some singular rank one perturbation.

(2) Suppose that  $\zeta - \Theta \notin H^2$  for any  $\zeta$  with  $|\zeta| = 1$ . We conclude that  $b \notin L^2(\mu)$  (otherwise, it would follow from (0.9) and (0.10) that for some  $\zeta$  of modulus one,  $y^{-1}(\zeta + \Theta(iy))(\zeta - \Theta(iy))^{-1} \rightarrow 0$  as  $y \rightarrow +\infty$ , which by (2.7) gives that  $\zeta - \Theta \in H^2$ ). Thus, condition (A\*) is satisfied and  $\mathcal{L}^*$  is correctly defined.

(3) First we claim that in this case  $\varphi$  is outer in  $\mathbb{C}^+$ . Indeed, by the assumption,  $\varphi$  has no zeros in  $\mathbb{C}^+$ . If there were a representation  $\varphi(z) = \psi(z)e^{i\gamma z}$ , where  $\psi \in (z+i)H^2$  and  $\gamma > 0$ , then the function  $\psi(z)\frac{e^{i\gamma z}-1}{z}$  would belong to  $K_\Theta$  and vanish on  $\Lambda$ , a contradiction. Finally,  $\Theta\bar{\varphi}$  also is outer in  $\mathbb{C}^+$ , and so  $\Theta = \xi^2\varphi/\bar{\varphi}$  for a unimodular constant  $\xi$ . Indeed, if  $\Theta\bar{\varphi} = I\varphi$  for a (meromorphic) inner function  $I$ , then  $Ig$  also is in  $K_\Theta$ . If  $I$  has a Blaschke factor  $\frac{z-z_0}{\bar{z}-\bar{z}_0}$ , then the function  $gI\frac{z-\lambda_0}{z-z_0}$  belongs to  $K_\Theta$  and vanishes on  $\Lambda$ , a contradiction. The case when  $I(z) = e^{i\gamma z}$  can be excluded as above.  $\square$

**2.4. Hermite–Biehler and Cartwright classes.** An entire function  $E$  is said to be in the *Hermite–Biehler class* (which we denote by  $HB$ ) if

$$|E(z)| > |E(\bar{z})|, \quad z \in \mathbb{C}^+.$$

We also always assume that  $E \neq 0$  on  $\mathbb{R}$ . For a detailed study of the Hermite–Biehler class see [49, Chapter VII]. Put  $E^*(z) = \overline{E(\bar{z})}$ . If  $E \in HB$ , then  $\Theta = E^*/E$  is an inner function

which is meromorphic in the whole plane  $\mathbb{C}$ ; moreover, any meromorphic inner function can be obtained in this way for some  $E \in HB$  (see, e.g., [35, Lemma 2.1]).

Given  $E \in HB$ , we can always write it as  $E = A - iB$ , where

$$A = \frac{E + E^*}{2}, \quad B = \frac{E^* - E}{2i}.$$

Then  $A, B$  are real on the real axis and have simple real zeros. Moreover, if  $\Theta = E^*/E$ , then  $2A = (1 + \Theta)E$ .

Any function  $E \in HB$  generates the *de Branges space*  $\mathcal{H}(E)$ , which consists of all entire functions  $f$  such that  $f/E$  and  $f^*/E$  belong to the Hardy space  $H^2$ , and  $\|f\|_E = \|f/E\|_{L^2(\mathbb{R})}$  (for the de Branges theory see [14]). It is easy to see that the mapping  $f \mapsto f/E$  is a unitary operator from  $\mathcal{H}(E)$  onto  $K_\Theta$  with  $\Theta = E^*/E$  (see, e.g., [35, Theorem 2.10]).

The reproducing kernel of the de Branges space  $\mathcal{H}(E)$  corresponding to the point  $w \in \mathbb{C}$  is given by

$$(2.13) \quad K_w(z) = \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\bar{w} - z)} = \frac{\overline{A(w)}B(z) - \overline{B(w)}A(z)}{\pi(z - \bar{w})}.$$

An entire function  $F$  is said to be of *Cartwright class* if it is of finite exponential type and

$$\int_{\mathbb{R}} \frac{\log^+ |F(x)|}{1 + x^2} dx < \infty.$$

For the theory of the Cartwright class we refer to [34, 46]. It is well-known that zeros  $z_n$  of a Cartwright class function  $F$  have a certain symmetry: in particular,

$$(2.14) \quad F(z) = Kz^m e^{icz} v.p. \prod_n \left(1 - \frac{z}{z_n}\right) \stackrel{\text{def}}{=} Kz^m e^{icz} \lim_{R \rightarrow \infty} \prod_{|z_n| \leq R} \left(1 - \frac{z}{z_n}\right),$$

where the infinite product converges in the “principal value” sense,  $c \in \mathbb{R}$  and  $K \in \mathbb{C}$  are some constants,  $m \in \mathbb{Z}_+$ .

A function  $f$  analytic in  $\mathbb{C}^+$  is said to be of bounded type, if  $f = g/h$  for some functions  $g, h \in H^\infty(\mathbb{C}^+)$ . If, moreover,  $h$  can be taken to be outer, we say that  $f$  is in the *Smirnov class* in  $\mathbb{C}^+$ . It is well known that if  $f$  is analytic in  $\mathbb{C}^+$  and  $\text{Im } f > 0$ , then  $f$  is in the Smirnov class [34, Part 2, Ch. 1, Sect. 5]. In particular, if  $t_n \in \mathbb{R}$ ,  $u_n > 0$  and  $\sum_n u_n < \infty$ , then the function  $\sum_n \frac{u_n}{t_n - z}$  is in the Smirnov class in  $\mathbb{C}^+$ . Consequently,  $\sum_n \frac{v_n}{t_n - z}$  is in the Smirnov class in  $\mathbb{C}^+$  for any  $\{v_n\} \in \ell^1$ .

Given a nonnegative function  $m$  on  $\mathbb{R}$  such that  $\log m \in L^1(dx/(x^2 + 1))$ , there is a unique outer function  $\mathcal{O}$  of Smirnov class, whose modulus is equal to  $m$  a.e. on  $\mathbb{R}$  and which satisfies  $\mathcal{O}(i) > 0$  (see [34, Part II, Chapter 3]). This function will be denoted by  $\mathcal{O}_m$ .

The following theorem due to M.G. Krein (see, e.g., [34, Part II, Chapter 1]) will be useful: *If an entire function  $F$  is of bounded type both in  $\mathbb{C}^+$  and in  $\mathbb{C}^-$ , then  $F$  is of Cartwright class. If, moreover,  $F$  is in the Smirnov class both in  $\mathbb{C}^+$  and in  $\mathbb{C}^-$ , then  $F$  is a Cartwright class function of zero exponential type.*

Finally, we remark that applications of de Branges spaces to the spectral theory of discrete selfadjoint operators and their perturbations are numerous and well-known, see for instance, [52] and also papers [58] and [75], that are closely related to our approach.

### 3. POSITIVE RESULTS ON COMPLETENESS OF $\mathcal{L}$ AND $\mathcal{L}^*$ FOR THE CASE OF SINGULAR RANK ONE PERTURBATIONS. PROOFS OF THEOREMS 0.1 AND 0.2

Let  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be a singular rank one perturbation of the multiplication operator  $\mathcal{A}$  defined in Section 0.2. We assume that  $\mu = \sum_n \mu_n \delta_{t_n}$  is a discrete measure. Then



Theorem 0.7 provides a model operator  $T$ , given in terms of meromorphic functions  $\Theta$  and  $\varphi$  satisfying condition (0.12).

We will assume that  $\varphi$  and  $\tilde{\varphi}$  have only simple zeros. In the case of multiple zeros, the root vectors, which were calculated in Lemma 2.4, should be taken into account. It can be checked that the same results hold in this case too and that, basically, the same arguments work.

The spectrum of  $\mathcal{A}$  is

$$\sigma(\mathcal{A}) = \{t_n\} = \{x \in \mathbb{R} : \Theta = -1\}.$$

We will use the notation  $a_n = a(t_n)$ ,  $b_n = b(t_n)$ . We recall that the measure  $\nu$ , defined in (2.8), is the Clark measure  $\sigma_{-1}$  for  $\Theta$ . In our case,

$$\nu = \sum_n \nu_n \delta_{t_n}, \quad \text{where } \nu_n = \mu_n |b_n|^2$$

(recall that  $b_n \neq 0$  for any  $n$ ), and  $\Theta'(t_n) = -2i/\nu_n$ . Finally, by (2.9),  $\varphi(t_n) = ia_n/b_n$ . Note that if  $a_n = 0$ , then  $\varphi(t_n) = 0$ .

**3.1. An abstract criterion for completeness of  $T^*$ .** Since  $\varphi$  is meromorphic and  $\frac{\varphi}{z+i}$  is in  $H^2$ , its Nevanlinna factorization in  $\mathbb{C}^+$  has the form

$$\varphi(z) = e^{i\alpha z} \cdot B(z) \cdot \mathcal{O}_{|\varphi|}(z),$$

where  $\mathcal{O}_{|\varphi|}$  is the outer part of  $\varphi$ ,  $B$  is a Blaschke product in  $\mathbb{C}^+$  and  $\alpha = \alpha(\varphi) \geq 0$ . The following proposition gives a criterion for the completeness of the eigenfunctions of  $T^*$ . This is a standard tool for the study of completeness for systems of reproducing kernels and many results of this type are known. For the case of the Paley–Wiener spaces it goes back to Levin [49, Appendix III, Theorem 6]. An explicit statement for the de Branges spaces is given in [32, Theorem 2.2]. We include a short proof of this statement.

**Proposition 3.1.** *Let  $\Theta$  and  $\varphi$  correspond to some rank one singular perturbation of a cyclic operator  $\mathcal{A}$  with discrete spectrum. More precisely, let  $\Theta$  be a meromorphic inner function and let  $\varphi \notin H^2$  satisfy (0.12).*

- (1) *If  $\alpha(\varphi) > 0$  or  $\alpha(\tilde{\varphi}) > 0$ , then the system  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$  is not complete in  $K_\Theta$ .*
- (2) *Suppose that  $\alpha(\varphi) = \alpha(\tilde{\varphi}) = 0$ . Then the following two statements are equivalent:*
  - (i) *the system  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$  is not complete in  $K_\Theta$ . (Thus, operator  $T^*$ , whenever it is correctly defined, is not complete.)*
  - (ii) *there exists a nonzero entire function  $F$  of zero exponential type, whose zeros lie in  $\mathbb{C}_- \cup \mathbb{R}$ , such that  $F\varphi \in H^2$  or  $F\tilde{\varphi} \in H^2$ .*

*Moreover, the function  $F$  in (ii) is always of Cartwright class.*

*Proof.* (1) Suppose that  $\alpha = \alpha(\varphi) > 0$ . Since  $\varphi$  is meromorphic, it has the form  $\varphi(z) = e^{i\alpha z} \varphi_1(z)$ , where  $\frac{\varphi_1}{z+i}$  is in  $H^2$ . Then the function  $\varphi_1(z) \frac{e^{i\alpha z} - 1}{z}$  belongs to  $K_\Theta$ , is non-zero and is orthogonal to the system  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$ , since the zeros sets of  $\varphi$  and  $\varphi_1$  (respectively, of  $\tilde{\varphi}$  and  $\tilde{\varphi}_1$ ) coincide. Analogously, if  $\tilde{\alpha} = \alpha(\tilde{\varphi}) > 0$ , then  $e^{-i\tilde{\alpha} z} \frac{\tilde{\varphi}}{z+i} \in H^2$  and, thus, the function  $\varphi(z) \frac{e^{i\tilde{\alpha} z} - 1}{z}$  is in  $K_\Theta$  and is orthogonal to the system  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$ .

(2) For the rest of the proof, we suppose that  $\alpha(\varphi) = \alpha(\tilde{\varphi}) = 0$ . Assume first that the system  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$  is not complete and so there is a nonzero  $g \in K_\Theta$  such that

$$(3.1) \quad \begin{aligned} 2\pi i g(\lambda) &= \langle g, k_\lambda \rangle = 0, \\ 2\pi i \tilde{g}(\lambda) &= \langle \tilde{k}_\lambda, g \rangle = 0, \end{aligned} \quad \lambda \in Z_\varphi.$$

We may write  $g = \varphi G$  for some function  $G$  which is meromorphic in  $\mathbb{C}$ , analytic in  $\mathbb{C}^+$  and on  $\mathbb{R}$ , and of bounded type in  $\mathbb{C}^+$ . Consider the function

$$\tilde{g}(z) = \Theta(z) \overline{g(\bar{z})} = \Theta(z) \overline{\varphi(\bar{z}) G(\bar{z})} = \tilde{\varphi}(z) G^*(z),$$

where  $G^*(z) = \overline{G(\bar{z})}$ . Since  $\tilde{g}$  vanishes at  $Z_{\tilde{\varphi}}$  we conclude that  $G^*$  has no poles in  $\mathbb{C}^+$  and thus  $G$  is an entire function. Moreover, since  $G = g/\varphi$  and  $G^* = \tilde{g}/\tilde{\varphi}$  in  $\mathbb{C}^+$  and  $\alpha(\varphi) = \alpha(\tilde{\varphi}) = 0$ , the functions  $G$  and  $G^*$  are in Smirnov class in  $\mathbb{C}^+$  and, by Krein's theorem,  $G$  is of zero exponential type and of Cartwright class. We have  $\varphi G = g \in H^2$  and  $\tilde{\varphi} G^* = \tilde{g} \in H^2$ .

Finally, to obtain from  $G$  the function  $F$  with zeros in  $\mathbb{C}^- \cup \mathbb{R}$  note that the zeros of  $G$  in  $\mathbb{C}^+$  satisfy the Blaschke condition. Let  $B$  be the Blaschke product constructed over  $Z_G \cap \mathbb{C}^+$  (counting multiplicities). Then  $F = G/B$  is an entire function with zeros in  $\mathbb{C}^- \cup \mathbb{R}$  which satisfies all the required properties (note that  $\varphi F = \varphi G/B \in H^2$ ). Analogously, we may move all zeros of  $G$  to the upper half-plane and then  $F^*$  will have all zeros in  $\mathbb{C}^- \cup \mathbb{R}$ .

To prove the converse, assume that there exists  $F$  as in (ii). Put  $g = F\varphi$ . By the assumption,  $f \in H^2$ , whereas  $\tilde{g}(z) = \tilde{\varphi}(z)F^*(z)$ . By the conditions on  $F$ , the ratio  $F^*/F$  is Blaschke product, while  $\tilde{\varphi}/\varphi$  is a ratio of two Blaschke products due to the condition  $\alpha(\varphi) = \alpha(\tilde{\varphi}) = 0$ . Hence,  $\tilde{\varphi}(z)F^*(z)$  is in  $H^2(\mathbb{C}^+)$  as soon as the function  $\varphi F$  belongs to this space. Thus  $\tilde{g}$  is in  $H^2$ , whence  $g \in K_\Theta$ . Since  $g$  vanishes on the set  $Z_\varphi$  and  $\tilde{g}$  on the set  $Z_{\tilde{\varphi}}$ , the function  $g$  is orthogonal to  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$  by (3.1). The case where we have a function  $F$  such that  $F\tilde{\varphi} \in H^2$  is treated analogously.  $\square$

**Remark.** It is an obvious, but useful observation that the function  $F$  in the statement 2 of Proposition 3.1 can be always chosen to have no zeros in  $\mathbb{C}^+$  (or in  $\mathbb{C}^-$ ). Moreover, it is not difficult to show that in the case of real type perturbations (i.e.,  $\varphi = \tilde{\varphi}$ ) the function  $F$  may be chosen to have *only real* zeros.

It would be interesting to compare the above theorem with [78, Theorem 5], where a completeness criterion is given in terms of Naboko's model of a nondissipative perturbation of a selfadjoint operator.

**3.2. Sufficient conditions for completeness.** The results stated in the Introduction show that completeness of  $T$  and  $T^*$  (equivalently,  $\mathcal{L}$  and  $\mathcal{L}^*$ ) are essentially different things. In this subsection we prove several results which show that, under some additional restrictions, both  $T$  and  $T^*$  are complete or the completeness of eigenfunctions of  $T^*$  (reproducing kernels  $k_\lambda$ ) implies the completeness of eigenfunctions of  $T$  (functions  $h_\lambda$ ).

We will need the following lemma.

**Lemma 3.2.** *Suppose that the meromorphic function  $\varphi$  associated with a perturbation  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  satisfies the conditions*

$$(3.2) \quad \sum_n \frac{|a_n b_n| \mu_n}{|t_n|} < \infty,$$

$$(3.3) \quad \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n} \neq \varkappa.$$

*Then  $|\varphi(iy)| \geq cy^{-1}$ ,  $|\tilde{\varphi}(iy)| \geq cy^{-1}$ ,  $y \rightarrow \infty$ , for some constant  $c > 0$ .*

*Proof.* By (0.11), we have

$$2\varphi(z) = (1 + \Theta(z)) \left( \varkappa - \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n} + \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n - z} \right).$$

Since

$$\sum_n \frac{a_n \bar{b}_n \mu_n}{t_n - iy} = \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n} \cdot \frac{t_n}{t_n - iy} \rightarrow 0, \quad y \rightarrow \infty,$$

(3.3) gives that  $|\varphi(iy)| \geq c_1 |1 + \Theta(iy)| > 0$ ,  $y \rightarrow \infty$ . It is easy to see that for any meromorphic inner function  $\Theta$  there exists a constant  $c_2 > 0$  such that for any  $\gamma \in \mathbb{C}$ ,  $|\gamma| = 1$ , and  $z \in \mathbb{C}^+$

$$(3.4) \quad |\gamma + \Theta(z)| \geq 1 - |\Theta(z)| \geq \frac{c_2 \operatorname{Im} z}{|z|^2 + 1}, \quad \operatorname{Im} z > 1$$

(for the proof note that if  $z_0$  is a zero of  $\Theta$ , then  $|\Theta(z)| \leq \left| \frac{z - z_0}{z - \bar{z}_0} \right|$  and if  $\Theta$  has no zeros, then  $\Theta(z) = \zeta e^{i\alpha z}$  for some  $\alpha > 0$ ,  $|\zeta| = 1$ ). Hence,  $|\varphi(iy)| \geq cy^{-1}$ ,  $y \rightarrow \infty$ , where  $c > 0$ . The same is true for  $\tilde{\varphi}$ , because it has the same representation as  $\varphi$  with conjugate coefficients.  $\square$

Next we give sufficient conditions for the joint completeness of  $\mathcal{L}$  and  $\mathcal{L}^*$ .

**Theorem 3.3.** *Let meromorphic  $\Theta$  and  $\varphi$  correspond to some singular rank one perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  such that the data  $(a, b, \varkappa)$  satisfy both (A) and (A\*), and let  $\alpha(\varphi) = \alpha(\tilde{\varphi}) = 0$ .*

(1) *Assume that at least one of the following two conditions holds:*

$$(3.5) \quad \limsup_{y \rightarrow \infty} y^N |\varphi(iy)| > 0$$

*for some  $N > 0$ , or*

$$(3.6) \quad \int_{\mathbb{R}} \frac{dt}{|\varphi(t + i\eta)|^\tau (1 + |t|)^N} < \infty$$

*for some  $N > 0$ ,  $\tau > 0$  and  $\eta \geq 0$ . Then both  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.*

(2) *(Generalized weak perturbations) If (3.2) and (3.3) hold, then the operator  $\mathcal{L}^*$  is correctly defined, and both  $\mathcal{L}$  and  $\mathcal{L}^*$  are complete.*

Notice that the case (2), which corresponds to rank one generalized weak perturbations (e.g., if  $a \in L^2(\mu)$  or  $b \in L^2(\mu)$ ) is, essentially, a particular case of Proposition 1.1. We will give a short direct proof, which does not use Macaev's theorem.

*Proof of Theorem 3.3.* Statement (2) is a special case of (1), because by Lemma 3.2 in this case (3.5) is satisfied. The proof of (1) will consist of several steps.

*Step 1: completeness of  $\mathcal{L}^*$ .* Let  $T$  be the model operator corresponding to  $\mathcal{L}$ . If the system of eigenfunctions of  $T^*$  is not complete, then, by Proposition 3.1, there exists a nonzero Cartwright class entire function  $F$  of zero exponential type with zeros in  $\mathbb{C}^- \cup \mathbb{R}$  such that  $F\varphi \in H^2$ . Hence,

$$|F(iy)\varphi(iy)| \leq Cy^{-1/2}, \quad y > 0.$$

Thus, if (3.5) holds, then  $\liminf_{y \rightarrow +\infty} y^{-N} |F(iy)| < +\infty$ . Since

$$F(z) = K \text{ v.p. } \prod_n \left(1 - \frac{z}{z_n}\right), \quad |F(iy)|^2 = |K|^2 \prod_n \frac{x_n^2 + (y + y_n)^2}{x_n^2 + y_n^2},$$

with  $z_n = x_n - iy_n$ ,  $y_n \geq 0$ , we conclude that  $F$  is a polynomial. Then  $\varphi \in H^2$ , which contradicts Theorem 0.7.

If (3.6) holds, then it follows from the Hölder inequality that

$$(3.7) \quad \int_{\mathbb{R}} |F(t + i\eta)|^\gamma (1 + |t|)^{-M} dt < \infty$$

for some  $\gamma \geq 0$  and  $M > 0$ . Since  $F$  is of zero type, we conclude again that  $F$  is a polynomial, a contradiction.

*Step 2: completeness of  $\mathcal{L}$  in the case  $a_n \neq 0$  for any  $n$ .* By Proposition 1.3,  $\mathcal{L}^* = \mathcal{L}_1$ , where we set  $\mathcal{L}_1 = \mathcal{L}(\mathcal{A}, b, a, \overline{\alpha})$ . If  $a$  is a cyclic vector for  $\mathcal{A}^{-1}$  (that is,  $a_n \neq 0$  for any  $n$ ), we may consider the model operator  $T_1$  associated with  $\mathcal{L}_1$ . Then  $\mathcal{L} = \mathcal{L}_1^*$  is complete if and only if  $T_1^*$  is complete. Note that in the corresponding pair  $(\Theta_1, \varphi_1)$  we have  $\varphi_1 = (1 + \Theta_1)\beta_1/2$ , where  $\beta_1$  is defined by (0.8) with the data  $(b, a, \overline{\alpha})$  in place of  $(a, b, \alpha)$ , and thus differs from  $\beta$  only by conjugation of the coefficients. Thus,  $\tilde{\varphi}_1 = (1 + \Theta_1)\beta/2$ . Now it follows from estimate (3.4) that if  $\varphi$  satisfies either (3.5) or (3.6) with  $\eta > 0$ , then

$$\limsup_{y \rightarrow \infty} y^N |\tilde{\varphi}_1(iy)| > 0 \quad \text{or} \quad \int_{\mathbb{R}} \frac{dt}{|\tilde{\varphi}_1(t + i\eta)|^\tau (1 + |t|)^{N+2}} < \infty$$

for some  $\tau > 0$ . If the eigenfunctions of  $T_1^*$  are not complete, then, by Proposition 3.1, there is a Cartwright class entire function  $F$  of zero exponential type with zeros in  $\mathbb{C}^- \cup \mathbb{R}$  such that  $F\tilde{\varphi}_1 \in H^2$  and we conclude, as in Step 1, that  $F$  is a polynomial, which is a contradiction because  $\tilde{\varphi}_1 \notin H^2$ .

The case  $\eta = 0$  is a bit more tricky. In this case we have

$$\int_{\mathbb{R}} \frac{dt}{|\varphi(t)|^\tau (1 + |t|)^N} = \int_{\mathbb{R}} \left| \frac{1 + \Theta_1(t)}{1 + \Theta(t)} \right|^\tau \cdot \frac{dt}{|\tilde{\varphi}_1(t)|^\tau (1 + |t|)^N} < \infty,$$

and we conclude that

$$\int_{\mathbb{R}} \left| \frac{1 + \Theta(t)}{1 + \Theta_1(t)} \right|^\gamma \frac{|F(t)|^\gamma}{(1 + |t|)^M} dt < \infty$$

for some  $\gamma > 0$  and  $M > 0$ . Since  $F$  is a Cartwright class function of zero type with zeros in  $\mathbb{C}^- \cup \mathbb{R}$ , we conclude that  $F$  is an outer function in  $\mathbb{C}^+$  and so is  $\frac{1+\Theta}{1+\Theta_1}F$ . Hence, we conclude that  $\frac{1}{(z+i)^M} \left( \frac{1+\Theta}{1+\Theta_1} \right)^\gamma F^\gamma \in H^1(\mathbb{C}^+)$ . Thus, we have also

$$(3.8) \quad \int_{\mathbb{R}} \left| \frac{1 + \Theta(t+i)}{1 + \Theta_1(t+i)} \right|^\gamma \frac{|F(t+i)|^\gamma}{(1 + |t|)^M} dt < \infty.$$

Applying again (3.4) we conclude that taking a larger  $M$  we may omit the factor  $\left| \frac{1+\Theta(t+i)}{1+\Theta_1(t+i)} \right|^\gamma$  in (3.8) and get (3.7). Thus,  $F$  is a polynomial, which once again gives a contradiction.

If  $a$  is not cyclic, our model does not formally apply to  $\tilde{\mathcal{L}}$ . We reduce the problem to the case where  $a$  is cyclic by considering the representation of the function  $\varphi$  with respect to a different Clark measure for the same space  $K_\Theta$ .

*Step 3: reduction to the case where  $a$  is cyclic.* We need to show that in conditions of the theorem the system  $\{h_\lambda\}_{\lambda \in Z_\varphi \cup \overline{Z}_\varphi}$  is complete in  $K_\Theta$ . Let  $\zeta \in \mathbb{C}$ ,  $|\zeta| = 1$ ,  $\zeta \neq -1$ , and let  $\sigma_\zeta$  be the corresponding Clark measure defined by (2.5). We may choose  $\zeta$  such that  $\zeta - \Theta \notin H^2$  and  $\Theta(0) \neq \zeta$  (thus,  $p(\zeta) = 0$ ). Since  $\Theta$  is meromorphic,  $\sigma_\zeta$  is a discrete measure,  $\sigma_\zeta = \sum_n \nu_n^\circ \delta_{t_n^\circ}$ , where  $\{t_n^\circ\} = \{t : \Theta(t) = \zeta\}$ , and we have, for some  $q^\circ \in \mathbb{R}$ ,

$$\frac{\zeta + \Theta(z)}{\zeta - \Theta(z)} = iq^\circ + \frac{1}{i} \sum_n \left( \frac{1}{t_n^\circ - z} - \frac{1}{t_n^\circ} \right) \nu_n^\circ, \quad z \in \mathbb{C}^+.$$

Since the function  $\varphi$  satisfies (0.12), we can represent it, as in the proof of Theorem 0.7,

$$2\varphi(z) = (1 - \zeta\Theta(z)) \left( \alpha^\circ + \sum_n \left( \frac{1}{t_n^\circ - z} - \frac{1}{t_n^\circ} \right) c_n^\circ \nu_n^\circ \right),$$

for some  $\varkappa^\circ$  and  $c_n^\circ$  satisfying  $\sum_n (t_n^\circ)^{-2} |c_n^\circ|^2 \nu_n^\circ < \infty$ . Let us write

$$\nu_n^\circ = |b_n^\circ|^2 \mu_n, \quad c_n^\circ = \frac{a_n^\circ}{b_n^\circ} \mu_n,$$

such a representation exists and is unique up to the choice of the arguments of  $b_n^\circ$ . Recall that by the properties of the Clark measures,  $\sum_n (t_n^\circ)^{-2} \nu_n^\circ < \infty$ . Then if we put  $a^\circ = (a_n^\circ)$ ,  $b^\circ = (b_n^\circ)$ , then  $a^\circ/x, b^\circ/x \in L^2(\mu^\circ)$ , where  $\mu^\circ = \sum_n \mu_n \delta_{t_n^\circ}$ .

Note, that  $\varphi(t_n^\circ) = ia_n^\circ/b_n^\circ$ . We may choose  $\zeta$  so that  $\varphi(t_n^\circ) \neq 0$  for all  $n$ . Indeed, we have a continuum of parameters  $\zeta$ , while the zeros of meromorphic function  $\varphi$  form a discrete set. Thus, with this choice of  $\zeta$  we have  $a_n^\circ \neq 0$  for all  $n$ .

At the same time, since  $\varphi$  satisfies (0.12), it follows from Proposition 2.2 (applied to the inner function  $\Theta^\circ = -\bar{\zeta}\Theta$  in place of  $\Theta$ ) that either  $a^\circ \notin L^2(\mu^\circ)$  or  $a^\circ \in L^2(\mu^\circ)$  and

$$(3.9) \quad \varkappa^\circ \neq \sum_n \frac{a_n^\circ \bar{b}_n^\circ \mu_n}{t_n^\circ}.$$

Thus the data  $(a^\circ, b^\circ, \varkappa^\circ)$  satisfy the condition (A) (with  $t_n^\circ$  in place of  $t_n$ ), whence the rank one perturbation  $\mathcal{L}^\circ = \mathcal{L}(\mathcal{A}^\circ, a^\circ, b^\circ, \varkappa^\circ)$  of the operator of the multiplication by  $x$  in  $L^2(\mu^\circ)$  is well defined. The pair  $(\Theta^\circ, \varphi)$  will be the model pair for  $\mathcal{L}^\circ$ . Assume for the moment that the data  $(a^\circ, b^\circ, \varkappa^\circ)$  also satisfy the condition (A\*) and so  $(\mathcal{L}^\circ)^*$  is well defined (we prove it in Step 4 below). Since  $\varphi$  satisfies (3.5) and (3.6), while now both  $a_n^\circ \neq 0$  and  $b_n^\circ \neq 0$  for all  $n$ , it follows from Step 2 that  $(\mathcal{L}^\circ)^*$  is complete, and so the system  $\{h_\lambda\}_{\lambda \in Z_\varphi \cup \bar{Z}_\varphi}$  is complete in  $K_{\Theta^\circ}$ . But for  $\Theta^\circ = -\bar{\zeta}\Theta$  we clearly have  $K_{\Theta^\circ} = K_\Theta$ .

*Step 4: the data  $(a^\circ, b^\circ, \varkappa^\circ)$  satisfy condition (A\*).* To complete the proof, it remains to verify (A\*) for the data  $(a^\circ, b^\circ, \varkappa^\circ)$ . If  $b^\circ \notin L^2(\mu^\circ)$ , then there is nothing to prove. Assume that  $b^\circ \in L^2(\mu^\circ)$ , which means that  $\sigma_\zeta(\mathbb{R}) = \sum_n \nu_n^\circ < \infty$ . It follows that there exists  $\xi$ ,  $|\xi| = 1$ , such that  $\xi - \Theta(iy) = O(y^{-1})$ ,  $y \rightarrow \infty$ , whence  $\xi - \Theta \in K_\Theta$  by (2.7). By the construction,  $\xi \neq -1$  and  $\xi \neq \zeta$ . Then it follows that  $\sum_n |b_n| \mu_n = \sigma_{-1}(\mathbb{R}) < \infty$  (indeed,  $\|\xi - \Theta\|_{L^2(\sigma_{-1})} = \|\xi + 1\|_{L^2(\sigma_{-1})} < \infty$ ). Since the data  $(a, b, \varkappa)$  satisfy (A\*) we have

$$\varkappa \neq \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n}.$$

Comparing two expansions for  $2\varphi$ ,

$$(1 + \Theta(iy)) \left( \varkappa - \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n} + \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n - iy} \right) = (1 - \bar{\zeta}\Theta(iy)) \left( \varkappa^\circ - \sum_n \frac{a_n^\circ \bar{b}_n^\circ \mu_n}{t_n^\circ} + \sum_n \frac{a_n^\circ \bar{b}_n^\circ \mu_n}{t_n^\circ - iy} \right),$$

as  $y \rightarrow \infty$  and using the fact that  $\Theta(iy) \rightarrow \xi$ , we conclude that (3.9) is satisfied (otherwise, the right-hand side would tend to 0 as  $y \rightarrow \infty$ , while the left-hand side has a nonzero finite limit).  $\square$

**Remarks.** 1. In [32, Theorem 2.4] completeness of  $\mathcal{L}$  was proved under the following assumptions on the function  $\varphi$ : there exists a weight  $w$  satisfying the Muckenhoupt  $(A_2)$  condition such that

$$|\varphi(t)| \leq w(t), \quad t \in \mathbb{R}, \quad \text{and} \quad \int_{\mathbb{R}} \frac{w(t)}{|\varphi(t)|^2(1+t^2)} dt < \infty.$$

Since  $\int_{\mathbb{R}} \frac{dt}{w(t)(1+t^2)} < \infty$  for any  $(A_2)$ -weight, already the second inequality implies  $\int_{\mathbb{R}} \frac{dt}{|\varphi(t)|(1+t^2)} < \infty$ . Thus, condition (3.6) of our Theorem 3.3 is much weaker than the conditions in [32, Theorem 2.4]. It is not so surprising, because the Muckenhoupt condition, which is intrinsic to the unconditional bases problem, seems to be too restrictive in



the completeness problems. Theorem 3.3 does not formally cover the case when  $(A^*)$  is not satisfied, but it is anyway obvious that in this case  $\mathcal{L}$  is not complete (indeed, we have  $a \notin L^2(\mu)$  and  $\text{Ker } \mathcal{L}^{-1} = \{0\}$ , while  $b \in \text{Ker}(\mathcal{L}^{-1})^* \neq \{0\}$ ).

2. Essentially, the argument of Step 3 reduces to considering perturbations  $\mathcal{A} \mapsto \mathcal{A} + \tau b \cdot b^* \mapsto A + (\tau b + a) \cdot b^*$  and choosing a parameter  $\tau \in \mathbb{R}$  such that the vector  $\tau b + a$  is cyclic for  $\mathcal{A}$ .

This section concludes with the proofs of Theorems 0.1 and 0.2. The proof of Theorem 0.2 will be based on the results of [8, Theorems 1.2 and 5.2].

*Proof of Theorem 0.1.* If  $\beta$  is given by (0.8), then

$$\text{Im } \beta(iy) = \sum_n \frac{y a_n \bar{b}_n \mu_n}{t_n^2 + y^2}.$$

If  $a_n \bar{b}_n \geq 0$  for all  $n$ , we have

$$\lim_{y \rightarrow \infty} y \text{Im } \beta(iy) = \sum_n a_n \bar{b}_n \mu_n \in (0, +\infty],$$

and therefore  $|\beta(iy)| \geq C y^{-1}$ ,  $y \rightarrow \infty$ . Hence, by (3.4),  $|\varphi(iy)| \geq C_1 y^{-2}$ ,  $y \rightarrow \infty$ , and so the system  $\{k_\lambda\}_{\lambda \in Z_\varphi \cup \overline{Z_\varphi}}$  is complete in  $K_\Theta$  by Theorem 3.3 (note that  $\tilde{\varphi} = \varphi$ ).

We show that in the general case  $|\varphi(iy)| \geq C y^{-N}$  for some  $N$ . To simplify the notations put  $u_n = a_n \bar{b}_n \mu_n$ . Since there is only finite number of negative terms  $u_n$  and an infinite number of positive terms, there exists a minimal nonnegative  $k_0$  such that

$$\sum_n u_n t_n^{2k} = 0, \quad 0 \leq k < k_0, \quad \text{while} \quad \sum_n u_n t_n^{2k_0} \neq 0$$

(it may happen that  $\sum_n u_n t_n^{2k_0} = +\infty$ ). Then

$$\frac{\text{Im } \beta(iy)}{y} = \sum_n \frac{u_n}{t_n^2 + y^2} = \frac{1}{y^2} \sum_n \frac{u_n}{1 + \frac{t_n^2}{y^2}} = \frac{(-1)^{k_0}}{y^{2k_0}} \sum_n \frac{u_n t_n^{2k_0}}{y^2 + t_n^2}$$

(in the last equality, the formula

$$\frac{1}{1+q} = \sum_{k=0}^{k_0-1} (-1)^k q^k + \frac{(-1)^{k_0} q^{k_0}}{1+q}$$

has been applied to  $q = t_n^2/y^2$ ). Since

$$\lim_{y \rightarrow +\infty} y^2 \sum_n \frac{u_n t_n^{2k_0}}{y^2 + t_n^2} = \sum_n u_n t_n^{2k_0} \in (-\infty, +\infty] \setminus \{0\},$$

we have  $|\text{Im } \beta(iy)| \geq C y^{-2k_0-1} > 0$  for  $y > y_0 > 0$ . A similar estimate holds for  $\varphi$  and we can apply Theorem 3.3.  $\square$

*Proof of Theorem 0.2.* By Proposition 1.3,  $\mathcal{L}^* = \mathcal{L}(\mathcal{A}, b, a, \overline{\alpha})$ . Note that, by (i) or (ii)  $a_n \neq 0$  for any  $n$ . Consider the model from Theorem 0.7 for this perturbation. The functions  $\Theta$  and  $\varphi$  are then defined by formulas (0.10) and (0.11) where the roles of  $a$  and  $b$  are interchanged. Completeness of  $\mathcal{L}$  means that the corresponding operator  $T^*$  is complete, and we need to show that completeness of the system  $\{k_\lambda\}_{\lambda \in Z_\varphi}$  implies the completeness of the system  $\{h_\lambda\}_{\lambda \in Z_\varphi}$ .

In the first case the measure  $\nu = \sum_n \nu_n \delta_{t_n}$ ,  $\nu_n = |a_n|^2 \mu_n$ , is the Clark measure for  $K_\Theta$ . Note that  $a \notin L^2(\mu)$  if and only if  $\nu(\mathbb{R}) = \infty$ . Now the statement follows immediately

from [8, Theorem 1.2]. Since  $\varphi(t_n) = b_n/a_n$ , the second condition means that  $|\varphi(t_n)| \leq C|t_n|^{-N}$  for some  $N > 0$ . Application of [8, Theorem 5.2] completes the proof.  $\square$

#### 4. NONCOMPLETE PERTURBATIONS. PROOF OF THEOREM 0.3.

This section is devoted to the proof of Theorem 0.3, which shows that for any spectrum there exist singular rank one perturbations such both  $\mathcal{L}$  and  $\mathcal{L}^*$  are correctly defined (actually, both  $a$  and  $b$  are not in  $L^2(\mu)$ ) and, moreover,  $\mathcal{L}^*$  is complete, while  $\mathcal{L}$  is not. The proof is based on an extension of an idea suggested by Yuri Belov in [8, Example 1.3].

It is much easier to see that for any cyclic selfadjoint operator  $\mathcal{A}$  with any spectrum  $\{t_n\}$  such that  $|t_n| \rightarrow \infty$ ,  $|n| \rightarrow \infty$ , there exists a real type rank one singular perturbation  $\mathcal{L}$  of  $\mathcal{A}$ , which is not complete. Indeed, it suffices to take a singular perturbation  $\mathcal{L}$  such that  $a, b$  are real,  $b \in L^2(\mu)$ ,  $a \notin L^2(\mu)$  and  $\int x^{-1}a(x)\bar{b}(x)d\mu(x) = -1$ . Then it is easy to see that the adjoint to the bounded operator  $\mathcal{L}^{-1}$  has non-trivial kernel, which implies the non-completeness of  $\mathcal{L}$ . In this example,  $(A^*)$  does not hold, so that  $\mathcal{L}^*$  is not correctly defined.

We will use the following lemma on entire functions.

**Lemma 4.1.** *For any sequence  $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ ,  $t_n \rightarrow \infty$ , there exists a function  $n : [0, \infty) \rightarrow [0, \infty)$ ,  $\log r = o(n(r))$ ,  $r \rightarrow \infty$ , with the following property: there does not exist a non-constant entire function  $U$  of order less than 1 with real zeros such that*

- (1)  $U(0) \neq 0$ ;  $\int_0^R \frac{n_U(r)}{r} dr = o\left(\int_0^R \frac{n(r)}{r} dr\right)$ ,  $R \rightarrow \infty$ ;
- (2) the sequence  $\{U(t_n)\}_{n \in \mathbb{N}}$  is bounded.

*Proof.* Choose a sequence  $\{x_k\}_{k \in \mathbb{N}}$ ,  $x_k > 0$ , with the following properties:  $x_1 = 2$ ,  $2x_k < x_{k+1}^{1/2}$  and each interval  $(2x_k, x_{k+1}^{1/2})$  contains at least one point from the sequence  $\{t_n\}$ . Then  $x_k \geq 2^{2^{k-1}}$ . Denote by  $n$  the counting function of the sequence  $\{x_k\}$ . Assume now that  $U$  satisfies (1), (2). Then, in particular,  $n_U(r) \leq n(er) \log(er) = o(\log^2 r)$ ,  $r \rightarrow \infty$ . Also, for infinitely many  $k$  we have  $[x_k, x_{k+1}] \cap Z_U = \emptyset$ . Fix any such  $k$  and let  $x \in (2x_k, x_{k+1}^{1/2})$ . Denote by  $\{u_n\}$  the set of zeros of  $U$ . Without loss of generality assume that  $U(0) = 1$ . Then we have

$$\log |U(x)| = \sum_n \log \left| 1 - \frac{x}{u_n} \right|.$$

Let us estimate the summands whose contribution is negative, that is, those with  $u_n > x_{k+1}$  (note that  $x > 2x_k$  and so  $x/u_n > 2$  for  $0 < u_n < x_k$ ):

$$\begin{aligned} \sum_{u_n > x_{k+1}} \log \left| 1 - \frac{x}{u_n} \right| &= \int_{x_{k+1}}^{\infty} \log \left| 1 - \frac{x}{r} \right| dn_U(r) \\ &= n_U(x_{k+1}) \log \left| 1 - \frac{x}{x_{k+1}} \right| - \int_{x_{k+1}}^{\infty} \frac{n_U(r)}{r(r-x)} dr = O(1), \end{aligned}$$

when  $k$  is sufficiently large, since  $x/x_{k+1} \leq x_{k+1}^{-1/2}$ . Thus,

$$\log |U(x)| = \sum_n \log \left| 1 - \frac{x}{u_n} \right| = \sum_{u_n < x_k} \log \left| 1 - \frac{x}{u_n} \right| + O(1) \rightarrow +\infty$$

as  $k \rightarrow \infty$ ,  $[x_k, x_{k+1}] \cap Z_U = \emptyset$  and  $x \in (2x_k, x_{k+1}^{1/2})$ . This contradicts condition (2), because the interval  $(2x_k, x_{k+1}^{1/2})$  contains points from the sequence  $\{t_n\}$ .  $\square$

*Proof of Theorem 0.3.* In view of Theorem 0.7, we may prove an analogous statement in the model space. Thus, we need to find an inner function  $\Theta$  and an outer function  $\varphi$  such that  $\Theta = \varphi/\bar{\varphi}$  on  $\mathbb{R}$ ,  $1 + \Theta \notin H^2$ ,  $\varphi \notin H^2$ ,  $\frac{\varphi}{z+i} \in H^2$ ,  $\{t \in \mathbb{R} : \Theta(t) = -1\} = \{t_n\}$  and the system  $\{h_\lambda\}_{\lambda \in Z_\varphi}$  of eigenfunctions of  $T$  is not complete in  $K_\Theta$ , while the system  $\{k_\lambda\}_{\lambda \in Z_\varphi}$  of eigenfunctions of  $T^*$  is complete in  $K_\Theta$ . Note also that  $\ker \mathcal{L}$  (or  $\ker \mathcal{L}^*$ ) is nontrivial if and only if  $\varkappa = 0$ , which is equivalent to  $\varphi(0) = 0$  by Proposition 2.2, (5).

The function  $\Theta$  in question is meromorphic and, in view of the discussion in Subsection 2.4 we may reformulate the problem in terms of de Branges spaces. Thus, we need to construct a space  $\mathcal{H}(E)$ ,  $E = A - iB$ , and an entire function  $g$  with the following properties:

- (i)  $A \notin \mathcal{H}(E)$ ;
- (ii)  $g$  is real on  $\mathbb{R}$ , has only simple real zeros and  $g(0) \neq 0$ ;
- (iii)  $g \notin \mathcal{H}(E)$ , but  $\frac{g(z)}{z-\lambda} \in \mathcal{H}(E)$ ,  $\lambda \in Z_g$ ;
- (iv) the system  $\left\{\frac{g(z)}{z-\lambda}\right\}_{\lambda \in Z_g}$  is not complete in  $\mathcal{H}(E)$  and its orthogonal complement is infinite-dimensional;
- (v) the system  $\{K_\lambda\}_{\lambda \in Z_g}$  of reproducing kernels (2.13) is complete in  $\mathcal{H}(E)$ .

If such  $E$  and  $g$  are constructed, it will remain to put  $\Theta = E^*/E$  and  $\varphi = g/E$ ; they will satisfy the properties listed above.

**Strategy of the proof.** Without loss of generality we may assume that  $\mathbb{N} \subset \mathcal{N}$  and  $t_n \rightarrow \infty$ ,  $n \rightarrow \infty$ . Assume also that  $|t_n| \geq 1$ ,  $n \in \mathcal{N}$ . Let us fix an entire function  $A$  which is real on  $\mathbb{R}$  and whose zero set is exactly  $\{t_n\}_{n \in \mathcal{N}}$ . Choose a subsequence  $\mathcal{N}_1 = \{n_k\}$  of  $\mathcal{N}$  such that  $t_{n_{k+1}} > 2t_{n_k} > 0$ . By Lemma 4.1, there exists a function  $n_0 : [1, \infty)$ , such that if  $U$  is an entire function of order less than one such that its zero counting function  $n_U$  satisfies  $\int_1^R \frac{n_U(r)}{r} dr = o\left(\int_1^R \frac{n_0(r)}{r} dr\right)$ ,  $R \rightarrow \infty$ , and  $\{U(t_{n_k})\}$  is bounded, then  $U$  is a constant.

The main step of the proof will be a construction of an entire function  $S$  of order zero such that  $\int_0^R \frac{n_S(r)}{r} dr = o\left(\int_0^R \frac{n_0(r)}{r} dr\right)$ ,  $R \rightarrow \infty$ , and of an entire function  $g$  with simple real zeros distinct from  $\{t_n\}$  that have the following properties:

- (a)  $g$  has the representation

$$(4.1) \quad \frac{g(z)}{A(z)} = \sum_{n \in \mathcal{N}} \frac{d_n}{z - t_n},$$

where  $d_n$  are real, nonzero and, for any  $N > 0$ ,

$$(4.2) \quad 2^{|n|}|d_n| = o(|t_n|^{-N}), \quad |n| \rightarrow \infty, \quad n \notin \mathcal{N}_1,$$

$$(4.3) \quad 2^k|d_{n_k}| = o(t_{n_k}^{-N}), \quad k \rightarrow \infty$$

(thus, the sum in the definition of  $g$  converges absolutely and uniformly on compact sets);

- (b)  $g$  admits the estimate

$$(4.4) \quad \frac{C_1}{|S(z)|} \cdot \left(\frac{|\operatorname{Im} z|}{|z|^2 + 1}\right)^2 \leq \left|\frac{g(z)}{A(z)}\right| \leq \frac{C_2}{|S(z)|} \cdot \left(\frac{|z|^2 + 1}{|\operatorname{Im} z|}\right)^2, \quad z \notin \mathbb{R},$$

for some constants  $C_1, C_2 > 0$ .

Once  $g$  has been constructed, we will define a de Branges space  $\mathcal{H}(E)$ , where  $E = A - iB$ ,  $A \notin \mathcal{H}(E)$  and prove that  $g$  and  $E$  satisfy the above properties (i)–(v).

**Construction of  $S$  and  $g$ .** Define entire functions  $A_0$  and  $B_0$  by the formulas

$$(4.5) \quad A_0(z) = \prod_k \left(1 - \frac{z}{t_{n_k}}\right), \quad \frac{B_0(z)}{A_0(z)} = \sum_k \frac{v_k}{t_{n_k} - z},$$

where  $v_k \in (1, 2)$  are chosen so that  $B_0(t_n) \neq 0$  for  $n \in \mathcal{N} \setminus \mathcal{N}_1$  and  $B_0(0) \neq 0$ . Thus  $B_0/A_0$  is a Herglotz function and each interval  $(t_{n_k}, t_{n_{k+1}})$  contains exactly one zero of  $B_0$  which we denote by  $s_k$ . Choose a subsequence  $\{s_{k_j}\}$  of  $\{s_k\}$  so sparse that the entire function

$$S(z) = \prod_j \left(1 - \frac{z}{s_{k_j}}\right)$$

satisfies  $\int_0^R \frac{n_S(r)}{r} dr = o\left(\int_0^R \frac{n_0(r)}{r} dr\right)$  as  $R \rightarrow \infty$ .

Since  $t_{n_{k+1}} > 2t_{n_k}$ , an easy estimate of the sum in the second equation in (4.5) shows that there exists  $\delta > 0$  such that  $\text{dist}(s_l, \{t_{n_k}\}) \geq \delta$ . Therefore, for any  $N > 0$ , we have  $t_{n_k}^N = O(|S(t_{n_k})|)$ ,  $k \rightarrow \infty$ , and, in particular,  $\sum_k |S(t_{n_k})|^{-1} < \infty$ . Hence, we have

$$(4.6) \quad \frac{B_0(z)}{S(z)A_0(z)} = \sum_k \frac{v_k}{S(t_{n_k})(t_{n_k} - z)} = \sum_k \frac{p_k}{t_{n_k} - z},$$

where  $p_k = \frac{v_k}{S(t_{n_k})}$ . Indeed, the residues in the right- and left-hand sides coincide and it is easily seen that their difference is an entire function of zero type (apply Krein's theorem) which tends to 0 along  $i\mathbb{R}$ , and thus is identically zero. Next, put

$$(4.7) \quad \gamma(z) = 1 + \sum_{n \in \mathcal{N} \setminus \mathcal{N}_1} \frac{q_n}{t_n - z},$$

where  $q_n > 0$  and  $\sum q_n < 1$  (thus,  $\gamma(0) \neq 0$ ). The size of  $q_n$  will be specified later.

Now we define the entire function  $g$  by

$$(4.8) \quad \frac{g(z)}{A(z)} = \frac{B_0(z)}{S(z)A_0(z)} \gamma(z).$$

It is real on  $\mathbb{R}$ , has only real zeros and  $g(0) \neq 0$ , since  $B_0(0)\gamma(0) \neq 0$ . Therefore (ii) holds. Note that in (4.6),  $|2^k p_k| = o(t_{n_k}^{-N})$  for any  $N > 0$ . We have

$$\begin{aligned} \frac{g(z)}{A(z)} &= \left( \sum_k \frac{p_k}{t_{n_k} - z} \right) \left( 1 + \sum_{n \in \mathcal{N} \setminus \mathcal{N}_1} \frac{q_n}{t_n - z} \right) \\ &= \sum_k \left( 1 - \sum_{n \in \mathcal{N} \setminus \mathcal{N}_1} \frac{q_n}{t_{n_k} - t_n} \right) \frac{p_k}{t_{n_k} - z} + \sum_{n \in \mathcal{N} \setminus \mathcal{N}_1} \left( \sum_k \frac{p_k}{t_{n_k} - t_n} \right) \frac{q_n}{t_n - z}. \end{aligned}$$

Now we specify the choice of  $q_n$  justifying the convergence in the last equation and the validity of the interchange of sums. Namely, we put

$$q_n = (2|t_n|)^{-|n|} \min_k |t_n - t_{n_k}|, \quad n \in \mathcal{N} \setminus \mathcal{N}_1.$$

Then it is clear that the function  $g/A$  has the representation (4.1) with the coefficients

$$\begin{aligned} d_{n_k} &= -p_k \left( 1 - \sum_{n \in \mathcal{N} \setminus \mathcal{N}_1} \frac{q_n}{t_{n_k} - t_n} \right), \\ d_n &= -q_n \sum_k \frac{p_k}{t_{n_k} - t_n}, \quad n \in \mathcal{N} \setminus \mathcal{N}_1, \end{aligned}$$

which have the properties (4.2)–(4.3). Note also that the coefficients at  $(z - t_n)^{-1}$  are non-zero for any  $n \notin \mathcal{N}_1$  by the smallness of  $q_n$ , while  $d_{n_k} \neq 0$  by the choice of  $v_k$ .

By an easy estimate of the Herglotz integral, for any function  $f$  with  $\text{Im } f > 0$  in  $\mathbb{C}^+$  we have

$$(4.9) \quad \frac{C|\text{Im } z|}{|z|^2 + 1} \leq |f(z)| \leq \frac{C'(|z|^2 + 1)}{|\text{Im } z|}, \quad z \notin \mathbb{R},$$

for some  $C, C' > 0$ . Since both  $B_0/A_0$  and  $\gamma$  are Herglotz functions, (4.8) implies the estimate (4.4).

**Choice of the weights and construction of the de Branges space  $\mathcal{H}(E)$ .** Choose another subsequence of indices  $\mathcal{N}_2 = \{m_j\} \subset \mathbb{N}$  so that  $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ ,  $t_{m_{j+1}} > 2t_{m_j}$ , which is so sparse that for the entire function

$$T(z) = \prod_j \left(1 - \frac{z}{t_{m_j}}\right)$$

we have  $|yT(iy)| = o(|S(iy)|)$ ,  $|y| \rightarrow \infty$  and

$$(4.10) \quad \sum_{n \notin \mathcal{N}_1} 2^{|n|} d_n^2 |T(t_n + i)|^2 < \infty, \quad \sum_k 2^k d_{n_k}^2 |T(t_{n_k} + i)|^2 < \infty.$$

This is possible by the properties (4.2)–(4.3) of the coefficients  $d_n$ .

Now we define the positive weights  $\nu_n$  as follows:

$$\begin{aligned} \nu_{n_k} &= d_{n_k}^2, & n_k &\in \mathcal{N}_1, \\ \nu_{m_j} &= 1, & m_j &\in \mathcal{N}_2, \\ \nu_n &= 2^{|n|} d_n^2, & n &\notin \mathcal{N}_1 \cup \mathcal{N}_2. \end{aligned}$$

By construction  $\sum_{n \in \mathcal{N}} \nu_n = \infty$ , but  $\sum_{n \in \mathcal{N}} t_n^{-1} \nu_n < \infty$ . Indeed,  $\sum_{n \notin \mathcal{N}_2} \nu_n < \infty$ , while  $\sum_j t_{m_j}^{-1} < \infty$ . Therefore, we may define an entire function  $B$  by

$$(4.11) \quad \frac{B(z)}{A(z)} = \sum_n \frac{\nu_n}{t_n - z}.$$

Since  $\operatorname{Im} \frac{B}{A} > 0$  in  $\mathbb{C}^+$ , we get that  $E = A - iB$  is a function in the Hermite–Biehler class. Let  $\Theta = E^*/\bar{E}$  be the corresponding meromorphic inner function. Then we have  $A = E(1 + \Theta)$  and

$$\frac{1 - \Theta(z)}{1 + \Theta(z)} = \frac{1}{i} \sum_n \frac{\nu_n}{t_n - z}.$$

Thus, by (2.7)  $1 + \Theta \notin H^2$  (equivalently,  $A \notin \mathcal{H}(E)$ ). This gives (i).

**Proof of (iii).** First we notice that  $g \notin \mathcal{H}(E)$ . Indeed, if  $g$  were an element of  $\mathcal{H}(E)$ , then from the fact that  $\nu = \sum_n \nu_n \delta_n$  is a Clark measure  $\sigma_{-1}$  for  $K_\Theta$ , we would get

$$\sum_n \frac{|g(t_n)|^2}{|E(t_n)|^2} \nu_n = \sum_n d_n^2 \frac{|A'(t_n)|^2}{|B(t_n)|^2} \nu_n < \infty.$$

By (4.11),  $B(t_n)/A'(t_n) = -\nu_n$ . Hence the latter sum equals to  $\sum_n \nu_n^{-1} d_n^2$ , which is infinite by the definition of  $\nu_n$ , a contradiction. Thus  $g \notin \mathcal{H}(E)$ .

On the other hand, if  $\lambda$  is a zero of  $g$  (notice that  $Z_g \cap \{t_n\} = \emptyset$ ), then  $\lambda \in \mathbb{R}$  and

$$\sum_n \frac{|g(t_n)|^2}{(t_n - \lambda)^2 |E(t_n)|^2} \nu_n = \sum_n \frac{d_n^2}{(t_n - \lambda)^2 \nu_n} < \infty.$$

Recall that

$$\|K_{t_n}\|_E^2 = \frac{|B(t_n)A'(t_n)|}{\pi}, \quad \text{whence} \quad \tilde{K}_{t_n}(z) = \frac{K_{t_n}(z)}{\|K_{t_n}\|_E} = -\nu_n^{1/2} \frac{\sqrt{\pi} \operatorname{sign} B(t_n) A(z)}{z - t_n}.$$

Therefore, the interpolation formula

$$\frac{g(z)}{(z - \lambda)A(z)} = \sum_{n \in N} \frac{d_n}{t_n - \lambda} \cdot \frac{1}{z - t_n}$$



(which clearly holds since the difference of the left- and right-hand parts is an entire function of zero type which tends to zero along the imaginary axis) can be rewritten as

$$g(z) = \sum_n \frac{d_n}{\nu_n(t_n - \lambda)} \cdot \frac{A(z)}{z - t_n} = -\pi^{-1/2} \sum_n \frac{d_n \operatorname{sign} B(t_n)}{\nu_n^{1/2}(t_n - \lambda)} \cdot \tilde{K}_{t_n}(z),$$

and thus  $\frac{g}{z-\lambda} \in \mathcal{H}(E)$  for any  $\lambda \in Z_g$ .

If we put  $\varphi = g/E$ , then  $\varphi$  and  $\Theta$  correspond, by Theorem 0.7, to an almost Hermitian singular rank one perturbation associated with certain  $\mu$ ,  $a$  and  $b$ . In particular, condition (A) is satisfied. On the other hand, by construction,  $\sum_n \nu_n = \sum_n |b_n|^2 \mu_n = \infty$ , thus  $b \notin L^2(\mu)$  and (A\*) is satisfied. Therefore, the adjoint operator is correctly defined.

**Proof of (iv).** First let us show that the system  $\left\{ \frac{g(z)}{z-\lambda} \right\}_{\lambda \in Z_g}$  is not complete in  $\mathcal{H}(E)$ . Consider the function

$$T_1(z) = \prod_j \left( 1 - \frac{z}{t_{m_j} + \delta_j} \right),$$

where  $\delta_j > 0$  are chosen to be so small that

$$\sum_j |T_1(t_{m_j})|^2 = \sum_j \nu_{m_j} |T_1(t_{m_j})|^2 < \infty.$$

Also, for sufficiently small  $\delta_j$  we have  $|T_1(t_n)| \leq |T_1(t_n + i)| \asymp |T(t_n + i)|$ , whence, by (4.10)

$$\sum_{n \notin \mathcal{N}_1 \cup \mathcal{N}_2} \nu_n |T_1(t_n)|^2 = \sum_{n \notin \mathcal{N}_1 \cup \mathcal{N}_2} 2^n d_n^2 |T_1(t_n)|^2 < \infty$$

and

$$\sum_k \nu_{n_k} |T_1(t_{n_k})|^2 = \sum_k d_{n_k}^2 |T_1(t_{n_k})|^2 < \infty.$$

Thus,  $\sum_n \nu_n |T_1(t_n)|^2 < \infty$ .

Now we show that the function  $T_1 g$  admits the following expansion:

$$(4.12) \quad \frac{T_1(z)g(z)}{A(z)} = \sum_n \frac{g(t_n)}{B(t_n)} \cdot \frac{c_n}{z - t_n} \nu_n^{1/2}$$

with some  $\{c_n\} \in \ell^2$ . Indeed, put

$$c_n = \frac{T_1(t_n)B(t_n)}{A'(t_n)\nu_n^{1/2}} = -T_1(t_n)\nu_n^{1/2}.$$

Then  $\{c_n\} \in \ell^2$ . Since  $\frac{g(z)}{z-\lambda} \in \mathcal{H}(E)$  and so  $\left\{ \frac{g(t_n)}{(t_n-\lambda)E(t_n)} \nu_n^{1/2} \right\} \in \ell^2$  for any  $\lambda \in Z_g$ , the series on the right-hand side of (4.12) converges uniformly on compact sets and it is easily seen that the residues on the left- and right-hand side of (4.12) coincide. Therefore,

$$H(z) = \frac{T_1(z)g(z)}{A(z)} - \sum_n \frac{g(t_n)}{B(t_n)} \cdot \frac{c_n}{z - t_n} \nu_n^{1/2}$$

is an entire function of zero exponential type and  $|H(iy)| \rightarrow 0$  as  $|y| \rightarrow 0$  (recall that  $y|T_1(iy)| = o(|S(iy)|)$  while  $|g(iy)|/|A(iy)| \leq C|y|/|S(iy)|$ ). Thus,  $H \equiv 0$ .

Now put  $z = \lambda$ ,  $\lambda \in Z_g$ , in (4.12). We have

$$\sum_n \frac{g(t_n)}{B(t_n)} \cdot \frac{c_n}{\lambda - t_n} \nu_n^{1/2} = 0, \quad \lambda \in Z_g,$$

which is equivalent to

$$\left\langle \frac{g(z)}{z-\lambda}, h \right\rangle = 0, \quad \lambda \in Z_g, \quad \text{where} \quad h = \sum_n \nu_n^{1/2} \frac{c_n}{B(t_n)} K_{t_n}.$$

Since  $\nu_n^{1/2} |B(t_n)|^{-1} K_{t_n}$  coincides with the normalized kernel  $\tilde{K}_{t_n}$  up to a constant factor, we have  $h \in \mathcal{H}(E)$ . Thus, the orthogonal complement to the system  $\frac{g(z)}{z-\lambda}$ ,  $\lambda \in Z_g$ , is nontrivial.

Moreover, the above argument works as well for the function  $T_1/P_m$ , where  $P_m(z) = (z - z_1) \dots (z - z_m)$  and  $z_1, \dots, z_m$  are the first  $m$  zeros of  $T_1$ . Clearly, the functions  $h_m$ , constructed with  $T_1/P_m$  in place of  $T_1$ , are linearly independent. Thus, the orthogonal complement to the system  $\frac{g(z)}{z-\lambda}$ ,  $\lambda \in Z_g$  is infinite-dimensional.

**Proof of (v).** Now we show that  $g$  is the generating function of a complete system of reproducing kernels. If the system  $\{K_\lambda\}_{\lambda \in Z_g}$  is not complete in  $\mathcal{H}(E)$ , then there exists a nonzero entire function  $U$  such that  $Ug \in \mathcal{H}(E)$ . We have  $gU/E = h \in H^2(\mathbb{C}^+)$ . Since  $g/A = (B_0/A_0) \cdot \gamma \cdot S^{-1}$ , where both  $A_0/B_0$  and  $\gamma$  are functions with positive imaginary part (and, thus, in the Smirnov class) and  $S$  is of zero order, we conclude that

$$U = h \cdot \frac{E}{A} \cdot \frac{A_0}{B_0} \cdot \gamma^{-1} S$$

is in the Smirnov class in the upper half-plane. Since  $gU^*$  is also in  $\mathcal{H}(E)$ , the same is true for  $U$  in the lower half-plane, and we conclude, by Krein's theorem, that  $U$  is of zero type.

Since  $h \in H^2(\mathbb{C}^+)$ , we have  $|h(z)| \leq C_1 (\operatorname{Im} z)^{-1/2}$ ,  $z \in \mathbb{C}^+$ . Also, by (3.4)  $|2A(z)|/|E(z)| = |1 + \Theta(z)| \geq C_2(|z|^2 + 1)^{-1} \operatorname{Im} z$ ,  $z \in \mathbb{C}^+$ . Applying (4.9) to  $B_0/A_0$  and to  $\gamma$  we conclude that

$$|U(z)| \leq C_3 \frac{(|z|^2 + 1)^3}{(\operatorname{Im} z)^{7/2}} |S(z)|, \quad z \notin \mathbb{R}.$$

We could have taken  $U$  such that  $U(0) = 1$ . Then, by the Jensen formula,

$$\begin{aligned} \int_0^R \frac{n_U(r)}{r} dr &= \frac{1}{2\pi} \int_0^{2\pi} \log |U(Re^{it})| dt \\ &\leq \log C_3 + \frac{1}{2\pi} \int_0^{2\pi} \log |S(Re^{it})| dt + 3 \log(R^2 + 1) - \frac{7}{4\pi} \int_0^{2\pi} \log R |\sin t| dt \\ &= \int_0^R \frac{n_S(r)}{r} dr + O(\log R) = o\left(\int_0^R \frac{n_0(r)}{r} dr\right), \quad R \rightarrow \infty. \end{aligned}$$

On the other hand, it follows from the inclusion  $gU \in \mathcal{H}(E)$  that

$$\sum_n \frac{|g(t_n)U(t_n)|^2}{|E(t_n)|^2} \nu_n < \infty.$$

Since for  $n = n_k$  we have  $|g(t_n)|^2 |E(t_n)|^{-2} \nu_n = 1$  (see the proof of (ii)), we conclude that  $\sum_k |U(t_{n_k})|^2 < \infty$ . Thus  $\{U(t_{n_k})\}$  is bounded, whence, by the choice of  $n_0$ ,  $U \equiv \text{const}$ . However,  $g \notin \mathcal{H}(E)$ , and so  $U \equiv 0$ . This contradiction proves that the system  $\{K_\lambda\}_{\lambda \in Z_g}$  is complete in  $\mathcal{H}(E)$ .  $\square$

## 5. COMPLETENESS OF RANK ONE PERTURBATIONS OF A COMPACT SELFADJOINT OPERATOR

In this section we consider the counterparts of our results for the case of bounded (regular) perturbations of a compact selfadjoint operator. Assume that  $\mu = \sum_n \mu_n \delta_{s_n}$  with  $s_n \rightarrow 0$ ,  $s_n \neq 0$ . We consider a bounded rank one perturbation  $\mathcal{L}_0 = \mathcal{A}_0 + ab^*$  with  $a, b \in L^2(\mu)$ , which is of real type (thus,  $a_n \bar{b}_n \in \mathbb{R}$ ). Moreover, let us assume that the functions  $a$  and  $b$  have additional smoothness near 0, namely,

$$(5.1) \quad \sum_n \frac{|a_n b_n| \mu_n}{|s_n|} < \infty.$$

A typical example when (5.1) is satisfied is when there is  $\alpha \in [0, 1]$  such that

$$(5.2) \quad a \in |x|^\alpha L^2(\mu), \quad b \in |x|^{1-\alpha} L^2(\mu),$$

which means that

$$\sum_n |a_n|^2 |s_n|^{-2\alpha} \mu_n < \infty, \quad \sum_n |b_n|^2 |s_n|^{2\alpha-2} \mu_n < \infty.$$

We also impose one more restriction: assume that

$$(5.3) \quad \sum_n a_n \bar{b}_n s_n^{-1} \mu_n \neq -1.$$

If (5.2) holds for  $\alpha = 1$ , the latter condition just means that the kernel of  $\mathcal{L}_0$  is trivial. Indeed, let  $zf + \langle f, b \rangle a = 0$ , where  $f \in L^2(\mu)$ ,  $f \neq 0$ . Then  $f = -cx^{-1}a$ ,  $c = \langle f, b \rangle$ . Hence,  $c = -c \langle x^{-1}a, b \rangle$  and since  $c \neq 0$ , we have  $\langle x^{-1}a, b \rangle = -1$  which is exactly the opposite statement to (5.3).

We have the following statement on the completeness of eigenvectors of  $\mathcal{L}_0$  and  $\mathcal{L}_0^*$ , which essentially coincides with the Macaev theorem (for our subclass), at least when (5.2) holds with  $\alpha = 1$ .

**Proposition 5.1.** *Let  $\mathcal{A}_0$  be a compact selfadjoint operator with simple spectrum,  $\ker \mathcal{A}_0 = 0$ , and let  $\mathcal{L}_0$  be its bounded rank one perturbation satisfying (5.1) and (5.3). Then  $\mathcal{L}_0$  and  $\mathcal{L}_0^*$  are complete.*

*Proof.* This follows immediately from Proposition 1.1. By (5.1)  $\mathcal{L}_0$  is a rank one generalized weak perturbation and (5.3) is exactly the required (one-dimensional) invertibility condition.  $\square$

Now we prove Corollary 0.4, which shows that for a wide range of spectra there exist rank one perturbations which are complete while their adjoints are not.

*Proof of Corollary 0.4.* Put  $t_n = s_n^{-1}$ . Then the unbounded operator  $\mathcal{A} = \mathcal{A}_0^{-1}$  is unitary equivalent to the multiplication by the independent variable in some  $L^2(\mu)$ ,  $\mu = \sum_n \mu_n \delta_{t_n}$ . Now, let  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$  be the singular rank one perturbation of  $\mathcal{A}$  constructed in Theorem 0.3. Recall that in this construction both  $a$  and  $b$  do not belong to  $L^2(\mu)$  and so conditions (A) and (A\*) are satisfied. Moreover, by construction,  $\varphi(0) \neq 0$  for the corresponding function  $\varphi$ , whence, by statement (5) in Proposition 2.2,  $\varkappa \neq 0$ . Then, by Proposition 1.5, the operator  $\mathcal{L}_0 \stackrel{\text{def}}{=} \mathcal{L}^{-1}$  is a bounded rank one perturbation of  $\mathcal{A}_0$ . By the construction,  $\mathcal{L}_0$  is complete while  $\mathcal{L}_0^*$  is not (notice that  $\mathcal{L}_0^* = (\mathcal{L}^*)^{-1}$ ).  $\square$

Next we prove Theorem 0.5. The proof combines a spectral synthesis criterion due to Markus with recent results from [9]. Let  $T$  be a compact operator with a trivial kernel and simple spectrum in a Hilbert space  $H$ , such that the sequence of its eigenvectors  $\{x_n\}_{n \in I}$

is complete in  $H$  and minimal. Denote by  $\{x'_n\}_{n \in I}$  the system biorthogonal to  $\{x_n\}$ . By a theorem of Markus [56, Theorem 4.1],  $T$  admits the spectral synthesis if and only if the system  $\{x_n\}_{n \in I}$  is *strongly* or *hereditarily complete*, which means that for any partition  $I = I_1 \cup I_2$ ,  $I_1 \cap I_2 = \emptyset$ , the system  $\{x_n\}_{n \in I_1} \cup \{x'_n\}_{n \in I_2}$  is complete in  $H$ . For further properties and examples of hereditarily and nonhereditarily complete systems see [21].

*Proof of Theorem 0.5.* Put  $t_n = s_n^{-1}$ . Then the increasing sequence  $\{t_n\}$  satisfies

$$(5.4) \quad C|t_n|^{-N} \leq t_{n+1} - t_n = o(|t_n|), \quad |n| \rightarrow \infty.$$

for some  $C, N > 0$ . It was shown in [9, Theorem 1.6] that for any sequence satisfying (5.4) there exists a de Branges space  $\mathcal{H}(E)$  and an entire function  $g$  such that:

(i)  $A = \frac{E+E^*}{2}$  vanishes exactly on the set  $\{t_n\}$ , and  $E - e^{i\alpha}E^*$  is not in  $\mathcal{H}(E)$  for any real  $\alpha$ ;

(ii)  $g$  has only real zeros,  $g(0) \neq 0$ , and the systems  $\{K_\lambda\}_{\lambda \in Z_g}$  and  $\{\frac{g}{z-\lambda}\}_{\lambda \in Z_g}$  are complete in  $\mathcal{H}(E)$ , but not hereditarily complete. Here  $K_\lambda$  is the reproducing kernel of  $\mathcal{H}(E)$ .

If we put  $\Theta = E^*/E$  and  $\varphi = g/E$ , then the systems  $\{k_\lambda\}_{\lambda \in Z_g}$  and  $\{\frac{\varphi}{z-\lambda}\}_{\lambda \in Z_g}$  are minimal and complete in  $K_\Theta$ . Note also that  $\Theta - e^{i\alpha} \notin H^2$  for any  $\alpha \in \mathbb{R}$ . By Theorem 2.5, there exists an almost Hermitian singular rank one perturbation  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  of a cyclic selfadjoint operator  $\mathcal{A}$  with spectrum  $\{t_n\}$ .

Since  $\varphi(0) \neq 0$  we conclude that  $\varkappa \neq 0$  (see Proposition 2.2, (5)), and so, by Proposition 1.5, the systems  $\{k_\lambda\}_{\lambda \in Z_g}$  and  $\{\frac{\varphi}{z-\lambda}\}_{\lambda \in Z_g}$  are unitarily equivalent to the systems of eigenfunctions of some bounded rank one perturbation  $\mathcal{L}_0$  of the operator  $\mathcal{A}_0 = \mathcal{A}^{-1}$  and of its adjoint, respectively. Since these systems are not hereditarily complete, we conclude, applying the above-cited theorem of Markus, that  $\mathcal{L}_0$  does not admit spectral synthesis.  $\square$

At the same time we are able to show that under certain restrictions on a perturbation the spectral synthesis holds up to a finite-dimensional defect.

**Theorem 5.2.** *Let  $\mathcal{A}_0$  be a compact selfadjoint operator with simple spectrum  $\{s_n\}$ ,  $s_n \neq 0$  (i.e.,  $\mathcal{A}_0$  is the operator of multiplication by the independent variable in  $L^2(\mu)$ ,  $\mu = \sum_n \mu_n \delta_{s_n}$ ). Assume that  $\{s_n\}_{n \in I}$  is ordered so that  $s_n > 0$  and  $s_n$  decrease for  $n \geq 0$ , and  $s_n < 0$  and increase for  $n < 0$  and*

$$|s_{n+1} - s_n| \geq C_1 |s_n|^{N_1}$$

*for some  $C_1, N_1 > 0$ . Let  $\mathcal{L}_0 = \mathcal{A}_0 + a^\circ(b^\circ)^*$  be a bounded rank-one perturbation of  $\mathcal{A}_0$  such that  $a^\circ, b^\circ \notin xL^2(\mu)$  and  $|a_n^\circ|^2 \mu_n \geq C_2 |s_n|^{N_2}$  for some  $C_2, N_2 > 0$ . Assume that operator  $\mathcal{L}_0$  is complete and that all its eigenvalues are simple and non-zero. Denote by  $\{f_j\}_{j \in J}$  the eigenvectors of  $\mathcal{L}_0$ . Then for any  $\mathcal{L}_0$ -invariant subspace  $M$  we have*

$$\dim(M \ominus \overline{\text{Lin}}\{f_j : f_j \in M\}) < \infty,$$

*where the upper bound for the dimension depends only  $N_1$  and  $N_2$ .*

The proof will use the following lemma.

**Lemma 5.3.** *Let  $\Theta$  be a meromorphic inner function and let  $\nu = \sum_{n \in I} \nu_n \delta_{t_n}$  be its Clark measure  $\sigma_\alpha$  for some  $\alpha$  (we assume that  $\{t_n\}$  is an increasing sequence). The following statements are equivalent:*

- (1) *There exist  $C, N > 0$  such that  $|\Theta'(t)| \leq C(|t| + 1)^N$ ,  $t \in \mathbb{R}$ ;*
- (2) *There exist  $c, M > 0$  such that*

$$(5.5) \quad \nu_n \geq c(|t_n| + 1)^{-M}, \quad t_{n+1} - t_n \geq c(|t_n| + 1)^{-M}.$$

*Proof.* The implication (i) $\implies$ (ii) is obvious. Indeed,  $\nu_n = \pi/|\Theta'(t_n)|$ . If we denote by  $\psi$  a continuous increasing branch of the argument of  $\Theta$  on  $\mathbb{R}$ , then  $\Theta(t) = \exp(i\psi(t))$  and  $|\Theta'(t)| = \psi'(t)$ ,  $t \in \mathbb{R}$ . We have

$$\psi(t_{n+1}) - \psi(t_n) = \int_{t_n}^{t_{n+1}} \psi'(t) dt = \pi,$$

whence  $t_{n+1} - t_n \geq \pi C^{-1}(|t_n| + 1)^{-M}$ . Thus, (ii) holds with  $M = N$ .

(ii) $\implies$ (i) Without loss of generality we assume that  $\nu = \sigma_{-1}$ . Then, by (2.5),

$$|\Theta'(t)| = \left| i + \sum_n \nu_n \left( \frac{1}{t_n - t} - \frac{1}{t_n} \right) \right|^{-2} \sum_n \frac{2\nu_n}{(t_n - t)^2}.$$

We will prove the estimate for sufficiently large  $t > 0$ , the argument for  $t < 0$  is analogous. It follows from (5.5) that, for sufficiently large  $k > 0$  (and, thus  $t_k > 0$ ),  $\text{card}\{n : 0 < t_n \leq 2t_k\} = O(t_k^{M+1})$ . Applying this inequality, (5.5) and the fact that  $\sum_n \nu_n(t_n^2 + 1)^{-1} < \infty$ , it is easy to deduce that

$$|\Theta'(t)| \leq \left| \frac{\nu_k}{t_k - t} + O(t_k^{M_1}) \right|^{-2} \left( \frac{2\nu_k}{(t_k - t)^2} + O(t_k^{M_1}) \right)$$

for  $|t - t_k| \leq \frac{\varepsilon}{2} t_k^{-N}$ , where  $M_1$  is some positive constant (depending on  $M$ ) and  $k$  is sufficiently large. We can assume that there is some index  $k > 0$  such that  $t \in [t_k, (t_k + t_{k+1})/2]$ ; the case when  $t \in [(t_k + t_{k+1})/2, t_{k+1}]$  is treated similarly. Thus, we conclude that  $|\Theta'(t)| = O(t_k^{M_2})$  for  $|t - t_k| \leq |t_k|^{-M_3}$ , where  $M_3 > M$  is so large that  $t_k^{M_1} = o(\nu_k/|t_k - t|)$  for these values of  $t$ .

Finally, for  $|t_k|^{-M_3} \leq t - t_k \leq (t_{k+1} - t_k)/2$ , we use the estimate

$$|\Theta'(t)| \leq \sum_n \frac{2\nu_n}{(t_n - t)^2} \leq C_1 t_k^2 \frac{\text{card}\{n : 0 < t_n \leq 2t_k\}}{(t - t_k)^2} + C_2 \sum_{n: t_n \notin [0, 2t_k]} \frac{\nu_n}{t_n^2}$$

for some constants  $C_1, C_2 > 0$  independent of  $k$ . This completes the proof.  $\square$

*Proof of Theorem 5.2.* We have  $\mathcal{L}_0 f_j = \lambda_j f_j$ , where  $\{\lambda_j\}$  are distinct and non-zero. There is a family of eigenvectors  $g_j$  of  $\mathcal{L}_0^*$  with  $\mathcal{L}_0^* g_j = \bar{\lambda}_j g_j$ , which form a biorthogonal family to  $\{f_j\}_{j \in J}$ . Put  $J_1 = \{j : f_j \in M\}$  and  $J_2 = J \setminus J_1$ . Assume that  $f \in M$  and  $h$  is orthogonal to all  $f_j$  with  $j \in J_1$ . The proof of [56, Lemma 4.2] shows that whenever  $f_j \notin M$ , the corresponding eigenvalue  $\lambda_j$  does not belong to  $\sigma(\mathcal{L}|_M)$  and  $h$  is orthogonal to  $g_j$ . This implies that  $h$  is orthogonal to the system  $\{f_j\}_{j \in J_1} \cup \{g_j\}_{j \in J_2}$ . We will show that in our situation, the orthogonal complement to this system is always finite-dimensional (for any decomposition  $J = J_1 \cup J_2$ ).

Let us pass again to the unbounded inverses (note that  $\ker \mathcal{L}_0 = \ker \mathcal{L}_0^* = 0$  since  $a^\circ, b^\circ \notin xL^2(\mu)$ ). Namely, put  $t_n = s_n^{-1}$ ,  $a_n = t_n a_n^\circ$ ,  $b_n = t_n b_n^\circ$ . Note that  $\tilde{\mu} = \sum_n \mu_n \delta_{t_n}$ , and  $|a_n|^2 \mu_n \geq C_2 |t_n|^{N_2+1}$ . Since  $a, b \notin L^2(\tilde{\mu})$ , the singular perturbation  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, -1)$  of the operator  $\mathcal{A} = \mathcal{A}_0^{-1}$  is well-defined, and  $\mathcal{L}_0$  is its inverse. Theorem 0.2 implies that  $\mathcal{L}^*$  is also complete.

Now, let  $\Theta$  and  $\varphi$  correspond to  $\mathcal{L}^* = \mathcal{L}(\mathcal{A}, b, a, -1)$  by Theorem 0.7, that is,

$$\frac{1 - \Theta(z)}{1 + \Theta(z)} = \frac{1}{i} \sum_n |a_n|^2 \mu_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right).$$

By Lemma 5.3 (applied to  $\nu_n = |a_n|^2 \mu_n$ ),

$$|\Theta'(t)| \leq C(|t| + 1)^K, \quad t \in \mathbb{R}.$$

for some  $C, K > 0$  with  $K$  depending only on  $N_1$  and  $N_2$ .

Let  $E$  be a function in  $HB$  such that  $\Theta = E^*/E$ . The eigenfunctions of  $\mathcal{L}^*$  are given by  $\{k_\lambda\}_{\lambda \in Z_\varphi} \cup \{\tilde{k}_\lambda\}_{\lambda \in Z_{\tilde{\varphi}}}$ . This system transforms, after multiplication by  $E$ , into the system  $\{K_\lambda\}_{\lambda \in Z_\varphi \cup Z_{\tilde{\varphi}}}$  of the reproducing kernels of the space  $\mathcal{H}(E)$ . Thus, our problem reduces to the following: given a complete and minimal systems of reproducing kernels  $\{K_\lambda\}_{\lambda \in \Lambda}$  in  $\mathcal{H}(E)$  with the biorthogonal system  $\{G_\lambda\}$ , we need to show that for any partition  $\Lambda = \Lambda_1 \cup \Lambda_2$ , the orthogonal complement to the system  $\{K_\lambda\}_{\lambda \in \Lambda_1} \cup \{G_\lambda\}_{\lambda \in \Lambda_2}$  is finite-dimensional. For the case when  $|\Theta'(t)| = O(|t|^K)$ ,  $|t| \rightarrow \infty$ , this statement was proved in [9, Theorem 5.3] (with the upper bound for the dimension depending only on  $K$ ).  $\square$

Finally, we prove Theorem 0.6 about Volterra rank one perturbations.

*Proof of Theorem 0.6.* As in the proof of Corollary 0.4, we pass to the equivalent problem for a singular perturbation of an unbounded operator  $\mathcal{A}$  (which is unitary equivalent to  $\mathcal{A}_0^{-1}$ ) on  $L^2(\mu')$ , where  $\mu' = \sum_n \mu_n \delta_{t_n}$ ,  $t_n = s_n^{-1}$ , and  $a'_n = (\mathcal{A}_0^{-1}a)_n = a_n/s_n$ ,  $b'_n = b_n/s_n$ . Thus we need to find  $t_n$  with  $|t_n| \rightarrow \infty$ ,  $\varkappa \in \mathbb{R}$ , and  $\mu' = \sum_n \mu_n \delta_{t_n}$  such that for any  $\alpha_1$  and  $\alpha_2$  as above there exist  $a'_n, b'_n$  satisfying conditions (A), (A\*) and such that

$$(5.6) \quad \sum_n |a'_n|^2 |t_n|^{2\alpha_1-2} \mu_n < \infty, \quad \sum_n |b'_n|^2 |t_n|^{2\alpha_2-2} \mu_n < \infty$$

and the function

$$\varphi(z) = \frac{1 + \Theta(z)}{2} \cdot \left( \varkappa + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) a'_n \overline{b'_n} \mu_n \right)$$

has no zeros in  $\mathbb{C}$ . Here  $\Theta$  is the meromorphic function defined by the formula

$$\Theta(z) = \frac{\rho(z) - i}{\rho(z) + i}, \quad \rho(z) = \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b'_n|^2 \mu_n.$$

Note that if  $\varphi$  has no zeros, then, in particular,  $\varphi(0) \neq 0$ , whence  $\varkappa \neq 0$ . Therefore, we may apply Proposition 1.5 to get a bounded rank one perturbation  $\mathcal{L}_0$  of  $\mathcal{A}_0$ , which is a Volterra operator such that  $\ker \mathcal{L}_0 = \ker \mathcal{L}_0^* = 0$ .

We take the entire function  $A(z) = \cos(\pi\sqrt{z})$ . Its zeros are  $t_n = (n - \frac{1}{2})^2$ ,  $n = 1, 2, \dots$ . It is easy to see that

$$(5.7) \quad \beta(z) \stackrel{\text{def}}{=} \frac{1}{A(z)} = 1 + \sum_{n \in \mathbb{N}} \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) c_n \mu_n,$$

where  $\mu_n = 1$ ,

$$c_n = -\frac{1}{A'(t_n)} = \frac{2}{\pi} (-1)^{n+1} \left( n - \frac{1}{2} \right).$$

Let  $\alpha_1 + \alpha_2 = 1 - \varepsilon$ , where  $\varepsilon > 0$ . We put  $a'_n = n^{2-2\alpha_1-\frac{1}{2}-\varepsilon}$  and

$$b'_n = \frac{c_n}{a'_n}, \quad |b'_n| \asymp \frac{n}{n^{2-2\alpha_1-\frac{1}{2}-\varepsilon}} = n^{2-2\alpha_2-\frac{1}{2}-\varepsilon}.$$

Hence  $a'_n, b'_n$  satisfy (5.6), and also (0.8) holds, with  $\varkappa = 1$  and with  $a', b'$  in place of  $a, b$ . At the same time,  $a', b' \notin L^2(\mu)$ , and so (A) and (A\*) are satisfied. Since  $c_n \in \mathbb{R}$ , we have  $\varphi/\bar{\varphi} = \Theta$ . By Theorem 0.7,  $\varphi$  corresponds to a real type singular perturbation with the smoothness properties (5.6). Also, we have

$$\varphi = \frac{1 + \Theta}{2A}.$$

Since the zeros  $t_n$  of  $1 + \Theta$  are exactly the poles of  $1/A$ ,  $\varphi$  has no zeros in  $\text{clos } \mathbb{C}^+$ . Hence the spectrum of the singular perturbation  $\mathcal{L}(\mathcal{A}, a', b', 1)$  is empty.



□

**Remark.** This proof shows that in Theorem 0.6, the unperturbed operator  $\mathcal{A}_0$  can be chosen to belong to  $\mathfrak{S}_p$  for any  $p > 1/2$ .

## 6. APPENDIX: PROOFS OF PROPOSITIONS 1.2 AND 1.3 ON SINGULAR BALANCED RANK $n$ PERTURBATIONS

*Proof of Proposition 1.2.* First we prove assertion (2). To this end, assume that  $\mathcal{L}$  is a balanced rank  $n$  singular perturbation of  $\mathcal{A}$ . We divide our argument into several steps.

(1) Consider the linear manifold  $N$ , which is the projection of  $G(\mathcal{A}) \cap G(\mathcal{L})$  onto the first component of  $H \oplus H$ . Then

$$N = \{x \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{L}) : \mathcal{A}x = \mathcal{L}x\}$$

and

$$(6.1) \quad \dim \mathcal{D}(\mathcal{A})/N = \dim \mathcal{D}(\mathcal{L})/N = n.$$

Put  $\mathcal{A}^{-1}\mathcal{L} \stackrel{\text{def}}{=} I + K$ , so that  $K : \mathcal{D}(\mathcal{L}) \rightarrow H$ . One has

$$y \in N \iff (y \in \mathcal{D}(\mathcal{L}) \ \& \ \mathcal{A}^{-1}\mathcal{L}y = y) \iff (y \in \mathcal{D}(\mathcal{L}) \ \& \ Ky = 0).$$

By (6.1),  $\text{rank } K = n$ .

Next, there exists a factorization

$$K = \tilde{a}\tilde{K},$$

where

$$(6.2) \quad \tilde{K} : \mathcal{D}(\mathcal{L}) \rightarrow \mathbb{C}^n, \ \text{Ran } \tilde{K} = \mathbb{C}^n; \quad \tilde{a} : \mathbb{C}^n \rightarrow H, \ \ker \tilde{a} = 0.$$

Any  $y \in \mathcal{D}(\mathcal{L})$  can be represented as

$$(6.3) \quad y = \mathcal{A}^{-1}\mathcal{L}y - Ky = y_0 + \tilde{a}c, \quad \text{where } y_0 \stackrel{\text{def}}{=} \mathcal{A}^{-1}\mathcal{L}y \in \mathcal{D}(\mathcal{A}), \ c \stackrel{\text{def}}{=} -\tilde{K}y \in \mathbb{C}^n.$$

In particular,

$$\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{A}) + \text{Ran } \tilde{a}.$$

We put

$$a = \mathcal{A}\tilde{a} : \mathbb{C}^n \rightarrow \mathcal{A}H.$$

(2) Consider the linear manifold

$$\{(c, y_0) \in \mathbb{C}^n \oplus \mathcal{D}(\mathcal{A}) : \tilde{a}c + y_0 \in \mathcal{D}(\mathcal{L}), \ \mathcal{L}(\tilde{a}c + y_0) = \mathcal{A}y_0\}.$$

Since this linear manifold has finite codimension in  $\mathbb{C}^n \oplus \mathcal{D}(\mathcal{A})$ , it has a form

$$\{(c, y_0) \in \mathbb{C}^n \oplus \mathcal{D}(\mathcal{A}) : \varkappa c + b^*y_0 = 0\}$$

for some  $b^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{C}^m$  and some complex matrix  $\varkappa \in \mathbb{C}^{m \times n}$  (here  $m \geq 0$  is an integer). So, for any pair  $(c, y_0) \in \mathbb{C}^n \oplus \mathcal{D}(\mathcal{A})$ , one has

$$(6.4) \quad \left[ \tilde{a}c + y_0 \in \mathcal{D}(\mathcal{L}) \ \& \ \mathcal{L}(\tilde{a}c + y_0) = \mathcal{A}y_0 \right] \iff \varkappa c + b^*y_0 = 0.$$

By putting here  $c = 0$ , we get that for  $y_0 \in \mathcal{D}(\mathcal{A})$ , the condition  $y_0 \in N$  is equivalent to the condition  $b^*y_0 = 0$ . Hence

$$(6.5) \quad \text{rank } b^* = n.$$

(3) Take any  $y \in \mathcal{D}(\mathcal{L})$ , and define its decomposition  $y = y_0 + \tilde{a}c$  as in (6.3). For these  $y, y_0$  and  $c$ , both conditions of the left-hand side of the equivalence (6.4) hold. By (6.3), any  $c \in \mathbb{C}^n$  can appear in this way. Hence  $\text{Ran } \varkappa \subseteq \text{Ran } b^*$ .

By (6.5), there is a matrix  $\Phi \in \mathbb{C}^{n \times m}$  such that  $\Phi|_{\text{Ran } b^*}$  is an isomorphism. Then the condition  $\varkappa c + b^*y = 0$  is equivalent to  $\Phi \varkappa c + \Phi b^*y = 0$ , and we can replace the pair  $(\varkappa, b)$  with  $(\Phi \varkappa, \Phi b)$ . In other words, it can be assumed that  $m = n$ , so that  $\varkappa \in \mathbb{C}^{n \times n}$ . We assert that  $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ .

Let us check first that  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$  is correctly defined and  $\mathcal{L}(\mathcal{A}, a, b, \varkappa) \subseteq \mathcal{L}$ . To this end, take any  $y = y_0 + \mathcal{A}^{-1}ac \in \mathcal{D}(\mathcal{L}(\mathcal{A}, a, b, \varkappa))$ . Then  $\varkappa c + b^*y_0 = 0$ . By (6.4),  $y \in \mathcal{D}(\mathcal{L})$ , and  $y_0 = \mathcal{A}^{-1}\mathcal{L}y$ . This shows that  $y$  determines  $y_0$  uniquely (and therefore  $(A_n)$  holds), and that  $\mathcal{L}(\mathcal{A}, a, b, \varkappa) \subseteq \mathcal{L}$ .

Now we check that  $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ . Take any  $y \in \mathcal{D}(\mathcal{L})$ . By (6.4),  $c \in \mathbb{C}^n$ , and that  $\mathcal{L}y = \mathcal{A}y_0$ . By (6.4),  $y \in \mathcal{D}(\mathcal{L}(\mathcal{A}, a, b, \varkappa))$  and  $\mathcal{L}(\mathcal{A}, a, b, \varkappa)y = \mathcal{L}y$ .

We leave the proof of assertion (1) to the reader. It follows essentially the same type of arguments.  $\square$

*Proof of Proposition 1.3.* (1) We conserve the notation of the previous proof. It was shown there that the space

$$N = \{y_0 \in \mathcal{D}(\mathcal{A}) : b^*y_0 = 0\}$$

has codimension  $n$  in  $\mathcal{D}(\mathcal{A})$ . If we understand  $b$  as an operator  $b : \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{C}^n$ , then

$$\{f \in \mathcal{A}H : f \perp N\} = \text{Ran } b.$$

It follows that  $(\text{clos}_H N)^\perp = (\text{Ran } b) \cap H$ .

Since  $N \subset \mathcal{D}(\mathcal{L})$ , it follows that any vector  $v \in H$  which is orthogonal to  $\text{clos } \mathcal{D}(\mathcal{L})$ , should have the form  $v = bd$ , some  $d \in \mathbb{C}^n$  such that  $v \in H$ . So  $\mathcal{L}$  is densely defined if and only if the only vector of the form  $v = bd \in H$ ,  $d \in \mathbb{C}^n$  which is orthogonal to  $\mathcal{D}(\mathcal{L})$  corresponds to  $d = 0$ .

For any  $v = bd$  as above, orthogonal to  $\mathcal{D}(\mathcal{L})$ , and any  $y = y_0 + \mathcal{A}^{-1}ac \in \mathcal{D}(\mathcal{L})$

$$0 = \langle y, v \rangle = \langle b^*y_0, d \rangle + \langle c, a^*(\mathcal{A}^{*-1}bd) \rangle = \langle c, -\varkappa^*d + a^*(\mathcal{A}^{*-1}bd) \rangle.$$

It follows from the proof of Proposition 1.2 that here  $c \in \mathbb{C}^n$  can be arbitrary. Therefore  $\mathcal{L}$  is densely defined if and only if  $(A_n^*)$  holds.

(2) Now assume that both conditions  $(A_n)$ ,  $(A_n^*)$  hold. Then both operators  $\mathcal{L}^*$  and  $\mathcal{L}_* \stackrel{\text{def}}{=} \mathcal{L}(\mathcal{A}^*, b, a, \varkappa^*)$  are well-defined. We have show that  $\mathcal{L}_* = \mathcal{L}^*$ . Notice first that

$$(6.6) \quad \begin{aligned} \mathcal{D}(\mathcal{L}_*) &\stackrel{\text{def}}{=} \{u = u_0 + \mathcal{A}^{*-1}bd : \\ &\quad d \in \mathbb{C}^n, u_0 \in \mathcal{D}(\mathcal{A}^*), \varkappa^*d + a^*u_0 = 0\}; \\ \mathcal{L}_*u &\stackrel{\text{def}}{=} \mathcal{A}^*u_0, \quad y \in \mathcal{D}(\mathcal{L}). \end{aligned}$$

Now take any pair of vectors  $y = y_0 + \mathcal{A}^{-1}ac \in \mathcal{D}(\mathcal{L})$ ,  $u = u_0 + \mathcal{A}^{*-1}bd \in \mathcal{D}(\mathcal{L}_*)$ . Then  $\mathcal{L}y = \mathcal{A}y_0$ ,  $\mathcal{L}_*u = \mathcal{A}u_0$ . By (0.3), (6.6),

$$\langle \mathcal{L}y, u \rangle - \langle \mathcal{A}y_0, u_0 \rangle = \langle \mathcal{A}y_0, \mathcal{A}^{*-1}bd \rangle = \langle b^*y_0, d \rangle = -\langle \varkappa c, d \rangle.$$

Similarly,

$$\langle y, \mathcal{L}_*u \rangle - \langle y_0, \mathcal{A}^*u_0 \rangle = -\langle \varkappa c, d \rangle,$$

so that  $\langle \mathcal{L}y, u \rangle = \langle y, \mathcal{L}_*u \rangle$  for all vectors  $y \in \mathcal{D}(\mathcal{L})$ ,  $u \in \mathcal{D}(\mathcal{L}_*)$ . Hence  $\mathcal{L}_* \subset \mathcal{L}^*$ . It is easy to see that if  $R, S$  are subspaces of  $H$  and

$$\dim(R \cap S)/S = \dim(R \cap S)/R = n,$$

then the same relations hold for the orthogonal complements  $R^\perp$ ,  $S^\perp$ . Denote by  $G(\mathcal{L})$  the graph of  $\mathcal{L}$  and introduce a unitary operator  $W$  on  $H \oplus H$  by the formula  $W(x, y) = (-y, x)$ . It is well-known that  $(WG(\mathcal{L}))^\perp = G(\mathcal{L}^*)$ . It follows that both  $\mathcal{L}_*$  and  $\mathcal{L}^*$  are rank  $n$  balanced perturbations of  $\mathcal{A}^*$ . Since  $G(\mathcal{L}_*) \subset G(\mathcal{L}^*)$ , it follows that  $\mathcal{L}_* = \mathcal{L}^*$ .  $\square$

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